Classifying gapped quantum phases using Matrix Product States

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Introduction

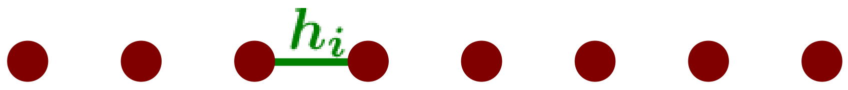
- What is the structure of gapped **one-dimensional quantum phases**?
- Is the **AKLT phase** different from e.g. a **dimerized system**,
  or a **trivial (product) phase**? (And if yes, in which sense?)
- Is there an analogy to **topological protection** in one dimension?
- This talk:

Classification of gapped 1D phases in the Matrix Product state formalism

- Structure of the talk:
  - what are Matrix Product States (MPS)
  - what do I mean by “phases”
  - phases in the MPS formalism
  - standard form of MPS for classification of phases
  - classification of gapped phases
  - classification of phases under symmetries

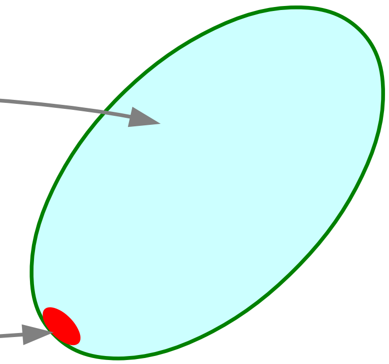
The area law

- $H = \sum_i h_i$ **local Hamiltonian**



  → can we **characterize the ground state** $|\Psi_0\rangle$?

- Problem: **exponentially large** Hilbert space!



- However: $H = \sum h_i$ has relatively **few parameters**
  $|\Psi_0\rangle$ lives in **small corner** of Hilbert space

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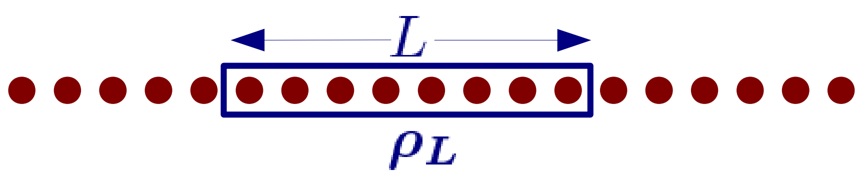
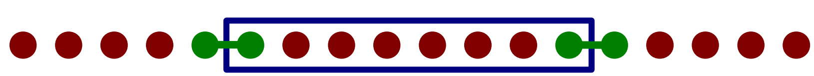
- **Guideline** for suitable **ansatz states**?
- **Area law** for ground states:

$$S(\rho_L) \sim \mathrm{const.}$$



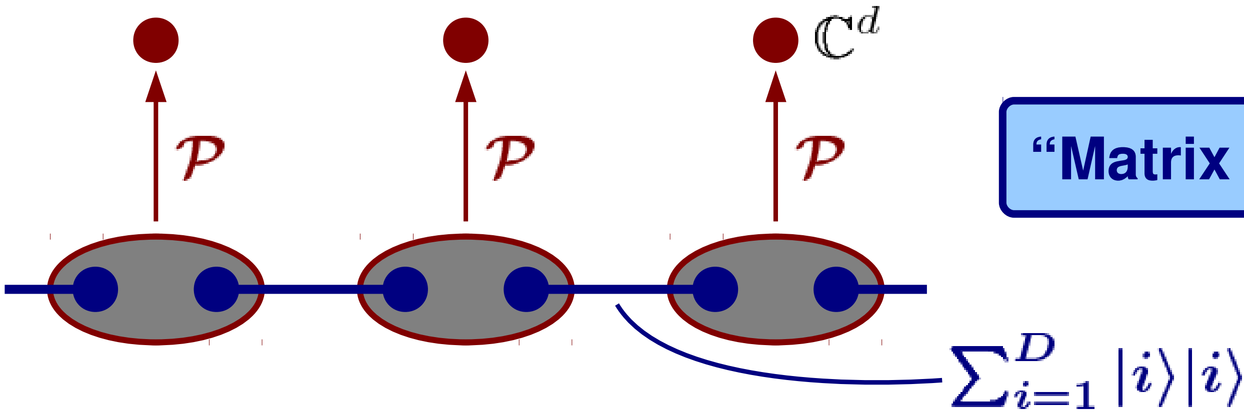
[Hastings, JSTAT '07]

- **entanglement** located around the **boundary**



Matrix Product States

- **Local description** of many-body states with **area law**?



"Matrix Product States" (MPS)

- system with **entropic area law** $\Leftrightarrow$ **well described** by MPS

[Verstraete & Cirac, PRB '06; Hastings, JSTAT '07; Schuch et al., PRL '08]

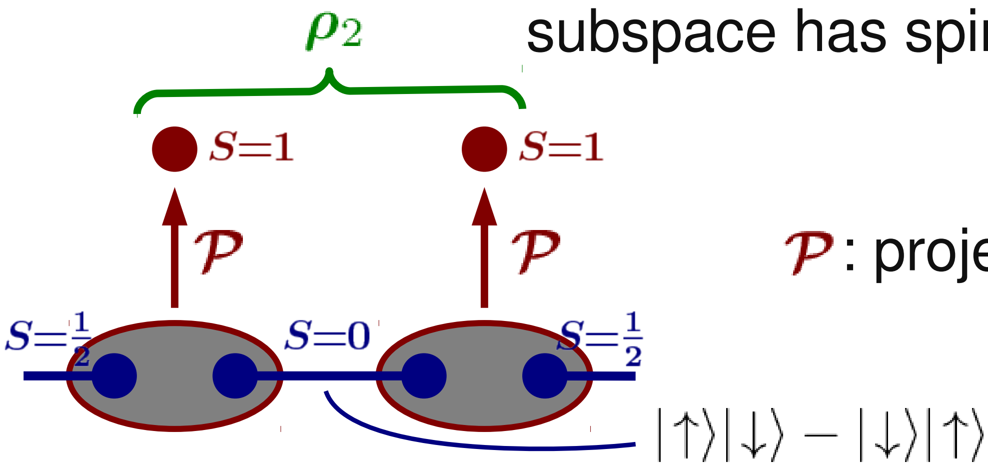
- describe **ground/thermal states** of **local Hamiltonians** efficiently

[Hastings, PRB '06, PRB '07, JSTAT '07]

- **toolbox** for **exactly solvable models**:
  - MPS are **exact ground states** of **local Hamiltonians**
  - many **properties** can be **computed analytically**

The AKLT state, and parent Hamiltonians

- Example: **AKLT state** [Affleck, Kennedy, Lieb & Tasaki, PRL '87]



subspace has spin $1 \otimes 1 = 0 \oplus 1 \oplus 2$ (2: **non-existent**)

$\mathcal{P}$: projector onto $S = 1$ subspace

- Hamiltonian: $H = \sum h_i$, with $\boldsymbol{h_i = \Pi_{S=2}}$ $\Rightarrow$ $h_i|\Psi_{\mathrm{AKLT}}\rangle = 0$

- $|\Psi_{\mathrm{AKLT}}\rangle$ is **unique ground state** of H, and H has a spectral gap

- Every MPS has an associated **parent Hamiltonian** with
  - unique ground state or fixed degeneracy
  - spectral gap [Fannes, Nachtergaele, Werner, CMP '92; Nachtergaele, CMP '96]

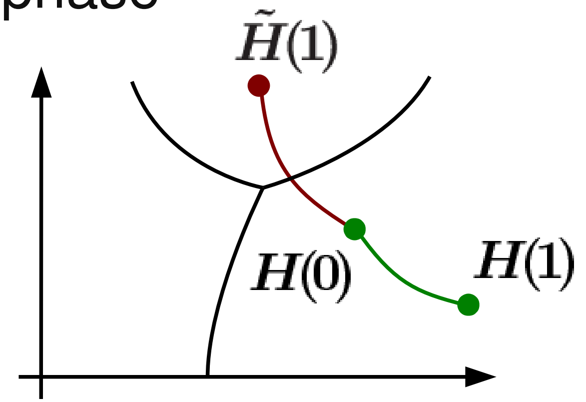
Framework for classification of phases

- we will study systems with **exact MPS ground states**
- same phase $\leftrightarrow$ we can **interpolate without phase transition**

- $H_0 = \sum_{i=1}^{N} h_0(i,i+1)\,,\ H_1 = \sum_{i=1}^{N} h_1(i,i+1)$ are in same phase

  iff there exists $H_\gamma = \sum h_\gamma(i,i+1)$ s.th.

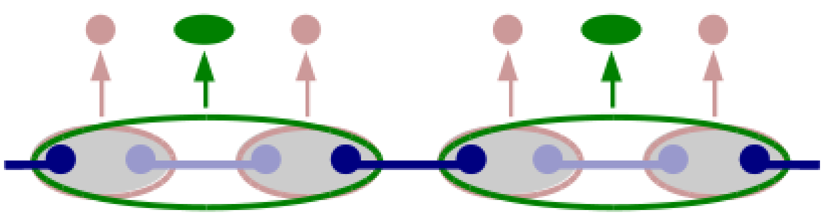
  - h_γ **continous** and **bounded**
  - H_γ is **uniformly gapped** in γ and N



- we allow **blocking**

  of a constant number of sites

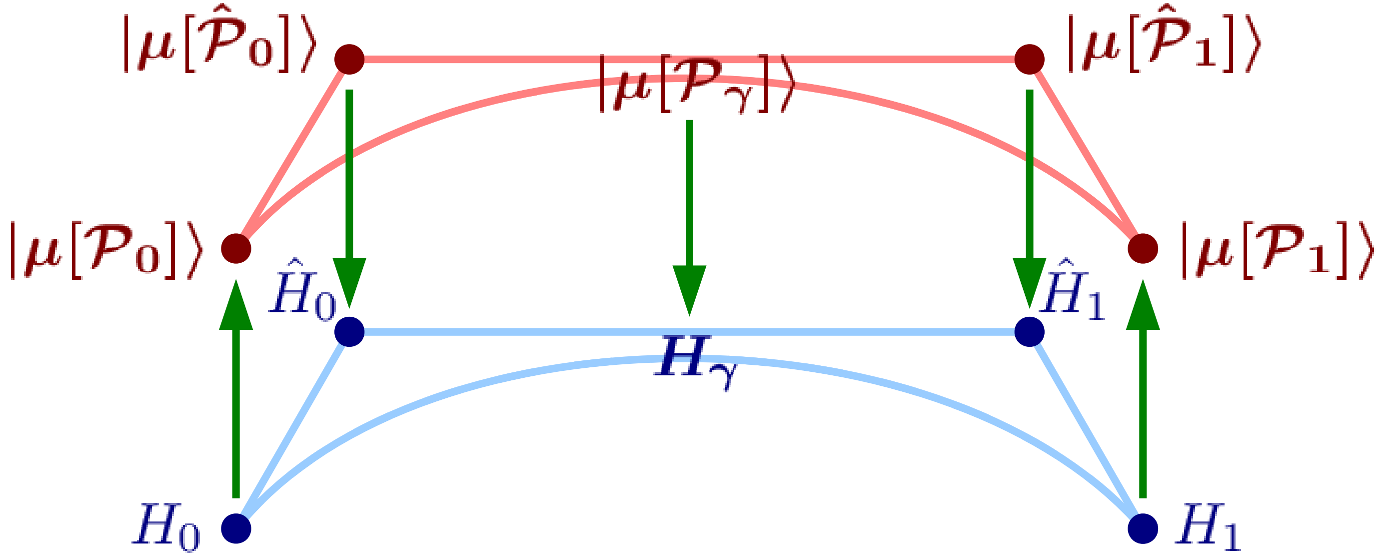
- we allow for **ancillas**



- extension: what if we impose **symmetries** $[H_\gamma, U_g^{\otimes N}] = 0$?

Classification using MPS

- state ↔ Hamiltonian duality: perform **classification for states**

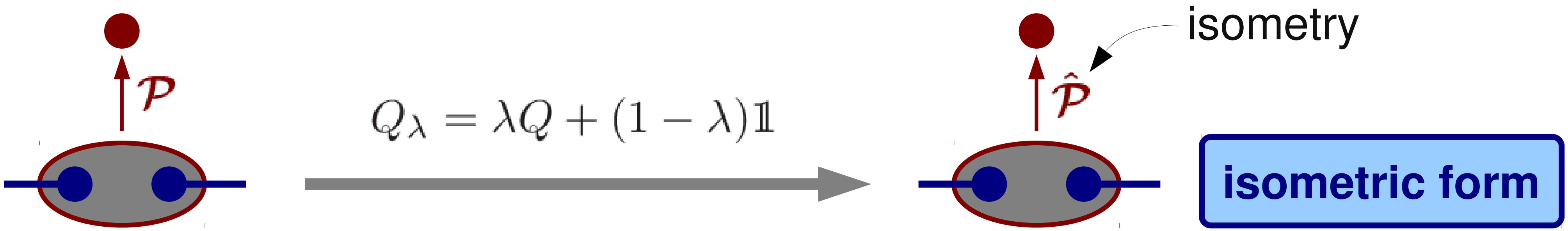


- construct interpolating path $\mathcal{P}_\gamma$ from $\mathcal{P}_0$ to $\mathcal{P}_1$

  → need to ensure **continuity** and **gappedness**!

- simplify interpolation using **normal form**
  - first interpolate to normal form (well-conditioned)
  - then interpolate between normal forms (simple structure)

The isometric form



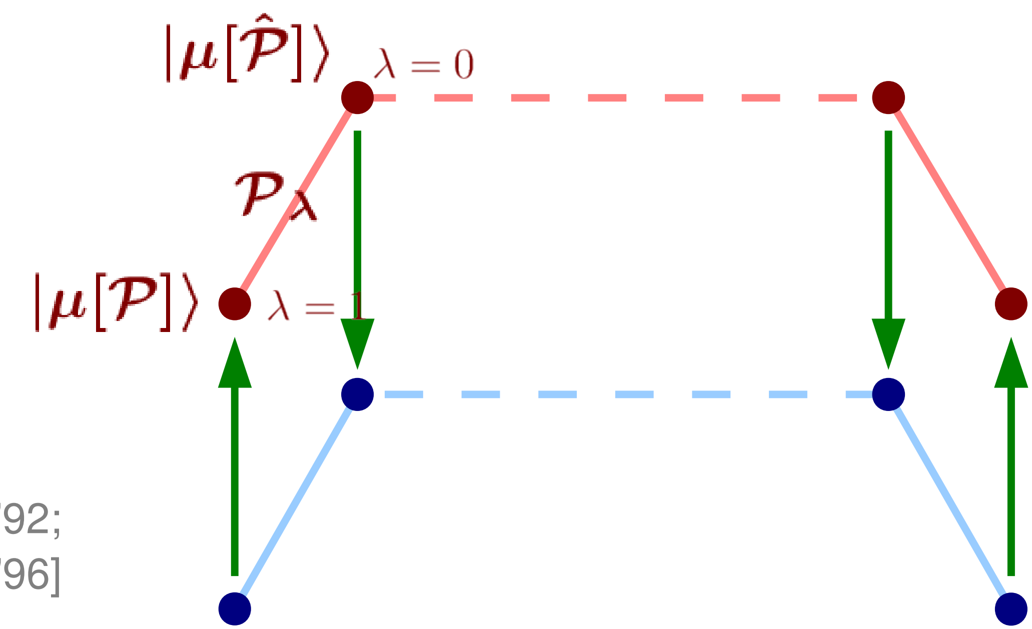
$\mathcal{P} = Q\,W$ $\qquad$ $\mathcal{P}_\lambda = Q_\lambda W$ $\qquad$ $\hat{\mathcal{P}} = W$

polar decomposition
$Q > 0,\ W$ isometry

- path of states $|\mu[\mathcal{P}_\lambda]\rangle = Q_\lambda^{\otimes N}|\mu[\hat{\mathcal{P}}]\rangle$
- **continuous** path of Hamiltonians

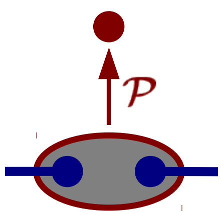
$$h_\lambda = (Q_\lambda^{-1})^{\otimes 2} h_0 (Q_\lambda^{-1})^{\otimes 2}$$

- Hamiltonians **uniformly gapped**
  follows from [Fannes, Nachtergaele, Werner, CMP '92; Nachtergaele, CMP '96]
- path **commutes with symmetry**
- classification reduces to **isometric form**



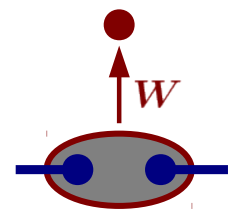
Classification – unique ground states

- systems with **unique ground state**:



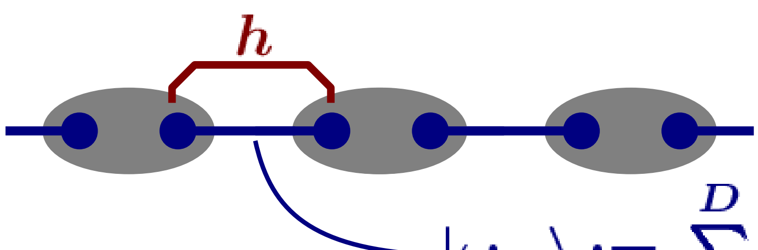
$\mathcal{P}$ **injective** (bijective)

isometric form →



W unitary

- **isometric form** (up to basis choice):



$$|\omega_D\rangle := \sum_{i=1}^{D} |i\rangle|i\rangle$$

$$h = \mathbb{1} - |\omega_D\rangle\langle\omega_D|$$

$\rightarrow$ characterized only by D

- interpolation between **different D and D'** :

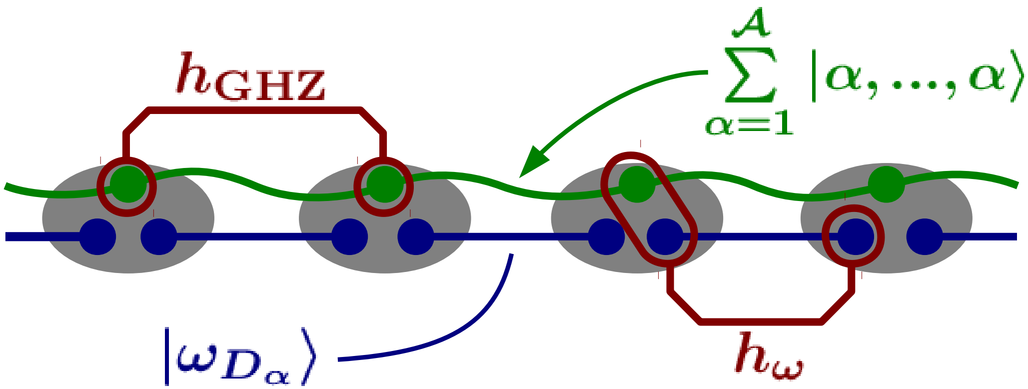
$$|\omega(\theta)\rangle := \theta|\omega_D\rangle + (1-\theta)|\omega_{D'}\rangle \qquad h(\theta) = 1 - |\omega(\theta)\rangle\langle\omega(\theta)|$$

$\Rightarrow$ **all states** in the **same phase**

Classification – degenerate ground states

- systems with $\mathcal{A}$-fold **degenerate ground state**

  $\rightarrow$ **isometric form** (up to basis choice):



$$h_{\mathrm{GHZ}} = \mathbb{1} - \sum_\alpha |\alpha,\alpha\rangle\langle\alpha,\alpha|$$

$$h_\omega = \sum_\alpha |\alpha\rangle\langle\alpha| \big(\mathbb{1} - |\omega_{D_\alpha}\rangle\langle\omega_{D_\alpha}|\big)$$

$\rightarrow$ commuting, since $|\alpha\rangle$ "classical"
(locally broken symmetry!)

- interpolation between **different** D_α **and** D'_α:

$$|\omega_\alpha(\theta)\rangle := \theta|\omega_{D_\alpha}\rangle + (1-\theta)|\omega_{D'_\alpha}\rangle$$

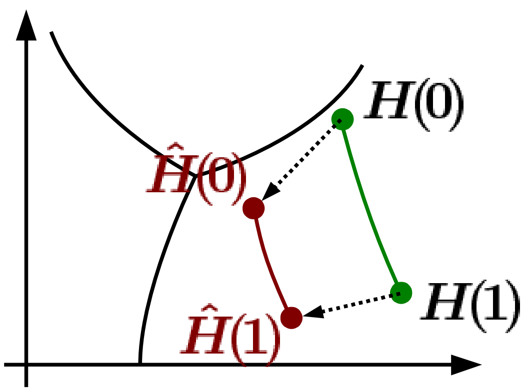
$$h_\omega(\theta) = \sum_\alpha |\alpha\rangle\langle\alpha| \big(\mathbb{1} - |\omega_\alpha(\theta)\rangle\langle\omega_\alpha(\theta)|\big)$$

$\Rightarrow$ all systems with **same** ground state **degeneracy** $\mathcal{A}$ in the **same phase**

Classification of phases without symmetries

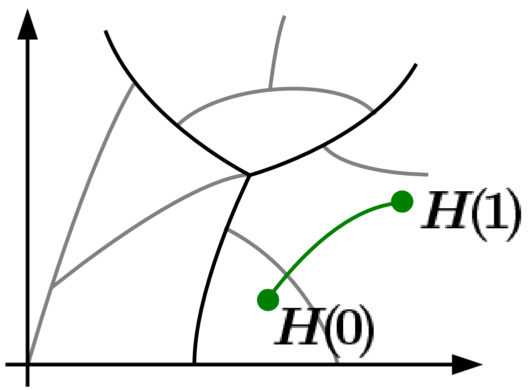
- Classification of **Hamiltonians** with **MPS ground states**
- H_1, H_2 in same phase $\leftrightarrow$ **smooth gapped path H_γ** exists

Systems with same ground state degeneracy $\mathcal{A}$ are in the same phase.
Different degeneracies label different phases.



- Proof steps:
  - MPS $\leftrightarrow$ **parent Hamiltonian** duality
  - construct **path of states** $|\mu[\mathcal{P}_\gamma]\rangle$ $\rightarrow$ induces path H_γ
  - **isometric form** $\mathcal{P} = QW \;\rightarrow\; \hat{\mathcal{P}} = W$ is in same phase
  - classify phases for isometric forms
  - isometric form $\Rightarrow$ **commuting parent Hamiltonian** $\Rightarrow$ simple class.

Phases under symmetries



- Phases under symmetries:

  Impose **constraint** $[H_\gamma, U_g^{\otimes N}] = 0$
  on interpolating path H_γ

- U_g : unitary representation of symmetry group G,
  $U_g U_h = U_{gh}$ [e.g. $G = \mathbb{Z}_2, G = \mathbb{Z}_2 \times \mathbb{Z}_2, G = SO(3)$]

- **Different representations** U_g^0, U_g^1 (e.g. spin-0 and spin-1 $\mathrm{SO}(3)$):

  $\rightarrow$ require invariance of H_γ under $U_g = U_g^0 \oplus U_g^1$

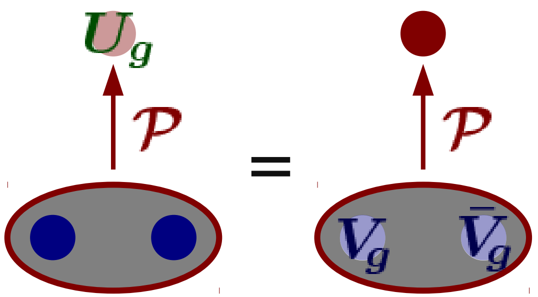
- symmetry of H_γ $\Leftrightarrow$ **symmetry** of corresponding **MPS** $|\mu[\mathcal{P}_\gamma]\rangle$

$$|\mu[\mathcal{P}_\gamma]\rangle = U_g^{\otimes N} |\mu[\mathcal{P}_\gamma]\rangle$$ (maybe with some phases …)

MPS and symmetries

- How is symmetry $|\mu[\mathcal{P}]\rangle = U_g^{\otimes N}|\mu[\mathcal{P}]\rangle$ **reflected in** $\mathcal{P}$?
- Restrict to **injective MPS**:

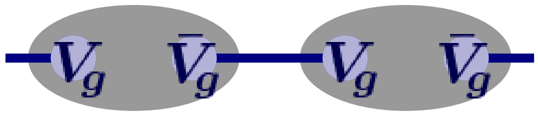
$$|\mu[\mathcal{P}]\rangle = U_g^{\otimes N}|\mu[\mathcal{P}]\rangle \;\Leftrightarrow\; U_g\mathcal{P} = \mathcal{P}(V_g \otimes \bar{V}_g)$$



- transformation to **isometric form**

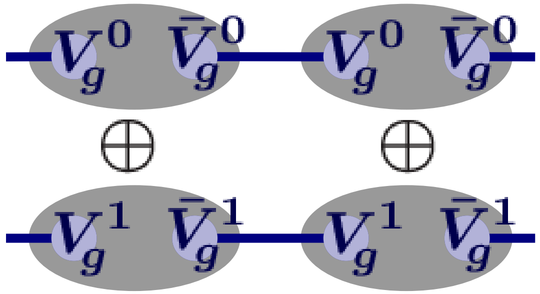
$P_\lambda = Q_\lambda W$ **keeps symmetry**

- **symmetry** action in **isometric form**:



in some **fixed basis**

- impose symmetry via $U_g = U_g^0 \oplus U_g^1$:
$\Rightarrow$ basis choice **unambiguous**



Projective representations

- What is the **structure** of $\boldsymbol{V_g}$?

$$\hat{U}_g = \mathcal{P}^{-1} U_g \mathcal{P} = V_g \otimes \bar{V}_g$$
$$\hat{U}_g \hat{U}_h = \hat{U}_{gh}$$

$\Rightarrow$ **projective representation**

$$\boldsymbol{V_g V_h = e^{i\omega(g,h)} V_{gh}}$$

- V_g only defined up to phase $V_g \leftrightarrow e^{i\phi_g} V_g$:

equivalence classes $\boldsymbol{\omega(g,h) \sim \omega(g,h) + \phi_{gh} - \phi_g - \phi_h}$

$\rightarrow$ equivalence classes form group: 2nd cohomology group $\mathrm{H}^2(G, \mathbb{C})$

Example 1: $G = \mathbb{Z}_2 \otimes \mathbb{Z}_2 = \{(0,0),(0,1),(1,0),(1,1)\}$

$V_{00} = \mathbb{1}$ | $V_{10} = Z$

$V_{01} = X$ | $V_{11} = Y$

$$\boldsymbol{V_{01} V_{10}} = XZ = -iY = \boldsymbol{-i\, V_{11}}$$
$$\boldsymbol{V_{10} V_{01}} = ZX = iY = \boldsymbol{i\, V_{11}}$$
$$\boldsymbol{V_{10} V_{01} V_{10}^\dagger V_{01}^\dagger} = XZXZ = -\mathbb{1}$$

Example 2:

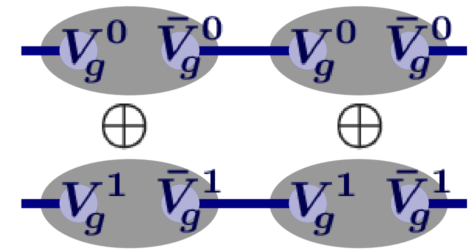
spin-$\frac{1}{2}$ repres. of $\mathrm{SO}(3)$:

$$\exp[2\pi i S_z] = -\mathbb{1}$$

- will show: **Equivalence class of $\boldsymbol{\omega}$ determines phase!** [cf. Pollmann et al., PRB '10; Chen, Gu, Wen, PRB '11]

Interpolation with same cohomology class

- Interpolate from $U_g^0 = V_g^0 \otimes \bar{V}_g^0$ to $U_g^1 = V_g^1 \otimes \bar{V}_g^1$

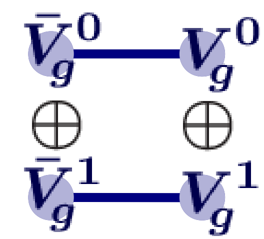


- Interpolate along path w/ **symmetry** $V_g \otimes \bar{V}_g$, where

[cf. Chen, Gu, Wen, PRB '11]

$$V_g = \left(\begin{array}{c|c} V_g^0 & \\ \hline & V_g^1 \end{array} \right)$$

with interpolating path $|\omega_\theta\rangle = \theta \sum_{i=1}^{D_0} |i,i\rangle + (1-\theta) \sum_{i=D_0+1}^{D_0+D_1} |i,i\rangle$



- key point: V_g is still **projective representation**:

$$V_g V_h = \begin{pmatrix} V_g^0 V_h^0 & \\ & V_g^1 V_h^1 \end{pmatrix} = \begin{pmatrix} e^{i\omega(g,h)} V_{gh}^0 & \\ & e^{i\omega(g,h)} V_{gh}^1 \end{pmatrix} = e^{i\omega(g,h)} V_{gh}$$

- Note: interpolation is in too big space: $(V_g^0 \oplus V_g^1) \otimes (\bar{V}_g^0 \oplus \bar{V}_g^1)$ instead of $(V_g^0 \otimes \bar{V}_g^0) \oplus (V_g^1 \otimes \bar{V}_g^1)$, but this can be fixed (using ancillas or blocking)

Separation of phases

- What if V_g^0 and V_g^1 belong to **different equivalence classes**?

- problem: $\boldsymbol{V_g}$ is **not a representation**:

$$\boldsymbol{V_g V_h} = \begin{pmatrix} V_g^0 V_h^0 & \\ & V_g^1 V_h^1 \end{pmatrix} = \begin{pmatrix} e^{i\omega_0(g,h)} V_{gh}^0 & \\ & e^{i\omega_1(g,h)} V_{gh}^1 \end{pmatrix} \neq \boldsymbol{e^{i\omega(g,h)} V_{gh}}$$

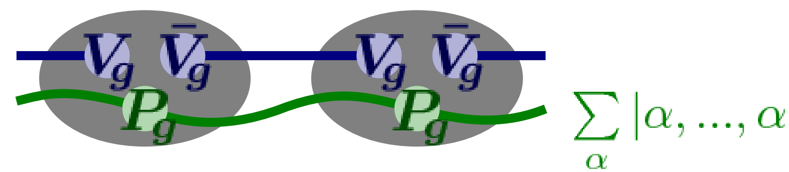
- how to prove **impossibility of interpolation**?

H_γ smooth & gapped → $|\mu[\mathcal{P}]\rangle$ smooth → $\mathcal{P}$ smooth → $V_g \otimes \bar{V}_g = \mathcal{P}^{-1} U_g \mathcal{P}$ smooth → V_g (up to phase) smooth → $\mathrm{d}\,\omega(g,h) = \delta_g + \delta_h - \delta_{gh}$ → **equivalence class of ω cannot change!**

Degenerate systems: Phases under symmetries

- What if system has **degenerate ground state**?
- Action of symmetry on virtual level:

$$U_g \mathcal{P} = \mathcal{P} P_g \Big(\bigoplus_\alpha V_g^\alpha \otimes \bar{V}_g^\alpha \Big)$$



> $\boldsymbol{P_g}$ **permutes** different **ground state sectors** $\boldsymbol{\alpha}$
>
> $\boldsymbol{V_g^\alpha}$ **induced representation** from **projective representation** $V_h^{\alpha_0}$ of the **subgroup** $\boldsymbol{G \supset H} = \{h \in G : \; P_g(\alpha_0) = \alpha_0\}$

- In addition to degeneracy:

Phases labelled by subgroup H and an element of $\mathrm{H}^2(H, \mathbb{C})$

= permutation action P_g

= equivalence class of projective representation $V_h^{\alpha_0}$

Summary

- classification of 1D systems with **exact Matrix Product ground states**
- **phases** defined by paths of **gapped Hamiltonians**
- MPS $\leftrightarrow$ **parent Hamiltonians**: construct **path of states**
- **Isometric form**:
  - same phase
  - captures **long-range properties**
  - **commuting** parent Hamiltonian
- Classification **without symmetries**:

Phases labelled by ground state degeneracy.

- Classification **with on-site symmetry** U_g :

Phases additionally labelled by
- **subgroup H of symmetry group**
- **equivalence classes of proj. representations of H**