Mardi 11/04/06

Aspects of the p-adic local Langlands programme, C. Breuil

Aspect I: A preliminary conjecture.

oral [I'm very glad to be in Harvard to talk about the p-adic local Langlands programme and I thank B. Mazur for giving me the opportunity to give this series of lectures.]

I am going to give 4 talks, on 4 aspects of the p-adic local Langlands programme. The motivation of these 4 talks is the following:

let $[L:\mathbb{Q}_p] < +\infty$ and $d \geqslant 1$ an integer, can one do:

$$\left\{\begin{array}{c}\text{p-adic Banach spaces +}\\ \text{continuous unitary}\\ \text{action of } GL_{d+1}(L)\end{array}\right\} \xleftarrow{\ ?\ } \left\{\begin{array}{c}d+1\text{- dim}^{\text{l}} \text{ p-adic representations}\\ \text{of } Gal(\overline{\mathbb{Q}}_p/L)\end{array}\right\}$$

[the reason one goes first this way is because many interesting Galois representations are known, namely the de Rham representations, and one would like to see their GL_{d+1}- "counterpart"].

Aspect I: When does an (irreducible) locally algebraic representation of $GL_{d+1}(L)$ admit an invariant norm? (ie $\|gv\| = \|v\|$)

Aspect II: (φ, Γ)- modules $(GL_2(\mathbb{Q}_p))$ (I and II already covered in Palo Alto)

Aspect III: Drinfeld spaces $(GL_2(\mathbb{Q}_p))$

Aspect IV: Mod p representations $(GL_2(\mathbb{Q}_{p^2}), GL_2(L))$.

The question in aspect I comes from the fact that if one starts with a de Rham representation, then it is easy to associate to it a locally algebraic representation of $GL_{d+1}(L)$. It is then hoped that the Banach space, or at least a J.H. component of it, will be obtained by completing this locally algebraic representation w.r. to a well chosen <u>invariant norm</u>. If there is no invariant norm, then we can forget this loc. alg. representation. So the question on invariant norms is a basic question if one is interested in de Rham represent$^{\underline{s}}$. Joint with <u>SCHNEIDER</u>.

In the rest of this talk, I fix K another finite extension of $\mathbb{Q}_p$ (the coefficients) and I assume $[L:\mathbb{Q}_p] = |\mathrm{Hom}_{\mathbb{Q}_p}(L, K)|$; $q = \#$ residue field of L, $|x|_L := q^{-val_L(x)}$, $val_L(\varpi_L) = 1$.

[

<u>Fontaine type categories</u>. I need a "Fontaine type" interpretation of Weil - Deligne representations.

I fix L' a finite Galois extension of L and I denote L'_0 its maximal unramified subfield. I will also assume $[L'_0 : \mathbb{Q}_p] = |\mathrm{Hom}_{\mathbb{Q}_p}(L'_0, K)|$.

Let me denote by $WD_{L'/L}$ the category of representations (r, N, V) of the Weil - Deligne group of L on a K-vector space V of finite dimension such that $r|_{W(\overline{\mathbb{Q}}_p/L')}$ is unramified.

Recall that the Weil group of L $=: W(\overline{\mathbb{Q}}_p/L)$ is the subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/L)$ of elements w mapping to an integral power $\alpha(w) \in \mathbb{Z}$ of the absolute arithmetic Frobenius in $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ and that a Weil - Deligne representation (r, N, V) on V is a map $r: W(\overline{\mathbb{Q}}_p/L) \longrightarrow \mathrm{Aut}_K(V)$ with open kernel together with a nilpotent K-linear endomorphism $N: V \longrightarrow V$ such that $r(w) N r(w)^{-1} = p^{\alpha(w)} N$.

Now, let me introduce a Fontaine type category: let $MOD_{L'/L}$ be the category of quadruples $(\varphi, N, \mathrm{Gal}(L'/L), D)$ where D is a free $L'_0 \otimes_{\mathbb{Q}_p} K$-module of finite rank endowed with:

$\varphi : D \longrightarrow D$ (Frobenius) bijective semi-linear $/L'_0$ $(\varphi(\lambda d) = \sigma(\lambda)\varphi(d)$

$N : D \to D$ (monodromy) linear s.t. $N\varphi = p\varphi N$ ($\Rightarrow$ nilpotent) ③

$\mathrm{Gal}(L'/L) \curvearrowright D$ semi-linear $/ L'_0$ $\big(g(\lambda d) = g(\lambda)\, g(d) \big)$

linear $/ K$

commuting with φ and N.

Fix an embedding $\sigma_0 : L'_0 \hookrightarrow K$, then Fontaine has defined a functor:

$$WD : \mathrm{MOD}_{L'/L} \longrightarrow WD_{L'/L} \quad \text{as follows:}$$

$$(\varphi, N, \mathrm{Gal}, D) \longmapsto (r, N, V) \text{ where:}$$

$$V := D \otimes_{L'_0 \otimes_{\mathbb{Q}_p} K,\ \sigma_0 \otimes \mathrm{Id}} K$$

$$N : V \to V \text{ is } N_D \otimes 1$$

$$r(w) : V \to V \text{ is } \underbrace{\bar{w}}_{\in\, \mathrm{Gal}(L'/L)} \circ\, \varphi^{-d(w)}$$

You can check that $r(w)\, N\, r(w)^{-1} = p^{d(w)} N$. Up to (non canonical) isomorphism (r, N, V) doesn't depend on σ_0).

<u>Lemma:</u> | The functor WD is an equivalence of categories.

The proof is left as an exercise. **Hint**: use the fact that [res. field of $L'_0 = \mathbb{F}_{p^f}$]

D can be written as $D = \bigoplus_{n=0}^{f-1} V_{\sigma_0 \circ \varphi_0^{-n}}$ where $V_{\sigma_0 \circ \varphi_0^{-n}} := D \otimes_{L'_0 \otimes K,\ \sigma_0 \circ \varphi_0^{-n} \otimes 1} K$

to go backwards and build D starting from V ($\varphi_0 =$ Frob on L'_0).

The lemma allows to see any WD representation as a "filtered module without the filtration".

<u>Local Langlands correspondence revisited.</u>

Recall that the local Langlands correspondence is a bijection:

$$\left\{\begin{array}{l}\text{isomorphism classes of}\\ \text{smooth irreducible repre-}\\ \text{-sentations of } GL_{d+1}(L)/\overline{\mathbb{Q}}_p\end{array}\right\} \xleftrightarrow{\text{Harris-Taylor, Henniart}} \left\{\begin{array}{l}\text{isomorphism classes of Weil-}\\ \text{Deligne representations } (r,N,V)\\ \text{over } \overline{\mathbb{Q}}_p \text{ such that } r \text{ is}\\ \text{semi-simple}\end{array}\right\}$$

satisfying lots of properties. Here, I choose the following normalization: if $\mathrm{rec} : W(\overline{\mathbb{Q}}_p/L)^{ab} \xrightarrow{\sim} L^\times$ is the reciprocity map sending the arithmetic Frobenius to the inverse of uniformizers, and if (r,N,V) is a Weil-Deligne representation as on RHS and π^{unit} the representation on LHS associated to (r,N,V), then:

$$\text{central char}\left(\pi^{\mathrm{unit}}\right) = \det(r,N,V)\circ \mathrm{rec}^{-1}.$$

I write now π^u for π^{unit}. Note that π^u depends on the choice of $q^{1/2}$.
In general, we are not going to work with the representation π^{unit}, however.
I want to define a representation π, a "better" representation.
Write $(r,N,V) = \bigoplus_i (r_i, N_i, V_i)$ with (r_i,N_i,V_i) indecomposable (all this over $\overline{\mathbb{Q}}_p$). Let π_i^u correspond to (r_i,N_i,V_i) by L.L.C. where π_i^u is a representation of GL_{d_i+1} for some d_i. Then π_i^u is called a "generalized Steinberg represent:". Then it is known that π^u is a quotient as follows:

normalized parabolic induction $\left\{ \mathrm{Ind}_P^{GL_{d+1}} \pi_1^u \otimes \cdots \otimes \pi_s^u \twoheadrightarrow \pi^u \right.$

[actually, one has to write the π_i^u in a certain order satisfying the so-called "does not precede" condition, then the parabolic induction doesn't depend on such an order]

I define $$\pi := \left(\mathrm{Ind}_P^{GL_{d+1}} \pi_1^u \otimes \cdots \otimes \pi_s^u\right) \otimes_{\overline{\mathbb{Q}}_p} |\det|_L^{-d/2}.$$

The following proposition follows from the Bernstein - Zelevinsky theory :

<u>Proposition</u>: Assume (r, N, V) is a representation on a K vector-space (i.e. V is a K-vector space), then π admits a unique model over K. Moreover, π doesn't depend on the choice of $q^{\frac{1}{2}}$.

still denoted π

<u>Example</u>: The typical example (and simplest example) is for $d = 1$ and

$$\pi^{\text{unit}} = |\cdot|_L \quad \longleftrightarrow \quad (r, N, V) = \begin{pmatrix} |\cdot|_L & 0 \\ 0 & 1 \end{pmatrix}$$

then $\pi = \mathrm{Ind}_{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}}^{GL_2(L)} |\cdot|_L \otimes |\cdot|_L^{-1}$ (here, the parabolic induction is NOT normalized)

<u>Conjecture</u>.

Fix
- $(r, N, V) \in WD_{L'/L}$ with r semi-simple
- for each $\sigma : L \hookrightarrow K$, integers $i_{j,\sigma} \in \mathbb{Z}$ such that:

$$i_{1,\sigma} < \cdots < i_{d+1,\sigma}.$$

Define ρ_σ = K-rational algebraic represent$^{\circ}$ of $GL_{d+1}(K)$ of highest weight:

$$-i_{d+1,\sigma} \leq -i_{d,\sigma} - 1 \leq \cdots \leq -i_{1,\sigma} - d \quad , \text{ i.e.}:$$

$$\rho_\sigma = \left(\mathrm{Ind}_{\begin{pmatrix} x_1 & & * \\ & \ddots & \\ 0 & & x_{d+1} \end{pmatrix}}^{GL_{d+1}(K)} x_1^{-i_{d+1,\sigma}} \otimes x_2^{-i_{d,\sigma}-1} \otimes \cdots \otimes x_{d+1}^{-i_{1,\sigma}-d} \right)^{alg}$$

$\shortparallel$ i.e. functions of $H^0(GL_{d+1}, \mathcal{O}_{GL_{d+1}})$

Let $\rho := \bigotimes_{\sigma : L \hookrightarrow K} \rho_\sigma$ with $GL_{d+1}(L)$ acting diagonally, $GL_{d+1}(L)$ acting on ρ_σ via the embedding $\sigma : GL_{d+1}(L) \hookrightarrow GL_{d+1}(K)$. Define π as above. So we have ρ, π, and we can consider $\rho \otimes_K \pi$. An invariant norm on $\rho \otimes_K \pi$ is by definition a p-adic norm $\| \ \|$ such that $\|g \cdot v\| = \|v\|$

<u>Conjecture</u>: The following conditions are equivalent:

(i) There is an invariant norm on $\rho \otimes_K \pi$

(ii) There is an object $(\varphi, N, \mathrm{Gal}(L'/L), D) \in \mathrm{MOD}_{L'/L}$ such that

$$WD(\varphi, N, \mathrm{Gal}(L'/L), D)^{F\text{-}ss} \simeq (r, N, V)$$

and a (weakly) admissible filtration preserved by $\mathrm{Gal}(L'/L)$ on $D_{L'} := L' \otimes_{L'_0} D = \prod_{\sigma : L \hookrightarrow K} D_{L'} \otimes_{L' \otimes_{\mathbb{Q}_p} K} (L' \otimes_{L,\sigma} K)$ such that:

$$\frac{\mathrm{Fil}^i D_{L',\sigma}}{\mathrm{Fil}^{i+1} D_{L',\sigma}} \neq 0 \iff i \in \{ i_{1,\sigma}, \ldots, i_{d+1,\sigma} \} \ (*) .$$

where $D_{L',\sigma} := D_{L'} \otimes_{L' \otimes_{\mathbb{Q}_p} K} L' \otimes_{L,\sigma} K$.

Transparency.

<u>Example 1</u>: • $L = L' = \mathbb{Q}_p$, $d = 1$, $N = 0$, r is unramified and given by arith. Frob. of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \longmapsto \begin{pmatrix} p^{\frac{k}{2}} & 0 \\ 0 & p^{\frac{k-2}{2}} \end{pmatrix}$ $\left(\text{ie} \begin{pmatrix} |\cdot|^{k/2} & 0 \\ 0 & |\cdot|^{\frac{k-2}{2}} \end{pmatrix} \right)$

(c.f. previous example where $\pi^u \neq \pi$)

$D = Ke_1 \oplus Ke_2$ • $i_1 = 1-k < i_2 = 0$ $k \geq 2$

$$\begin{cases} \varphi(e_1) = p^{-k/2} e_1 \\ \varphi(e_2) = p^{-\frac{k-2}{2}} e_2 \end{cases} \Downarrow \quad \rho \otimes_K \pi = \mathrm{Sym}^{k-2} K^2 \otimes_K \underbrace{\left(\mathrm{Ind}\, |\cdot| \otimes |\cdot|^{-1}\right) \otimes |\det|^{\frac{k-2}{2}}}_{\pi}$$

$\exists$ weakly admissible filtration $=$ $\begin{cases} \mathrm{Fil}^{-(k-1)} D = D \\ \mathrm{Fil}^{-(k-1)+1} D = \ldots = \mathrm{Fil}^0 D = K(e_1 + e_2) \end{cases}$

And one can prove there is an invariant norm on $\rho \otimes_K \pi$.

<u>Example 2</u>: • $L = L' = \mathbb{Q}_p$, $d = 1$, $N = 0$, r is unramified given by: arith. Frob. of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \longmapsto \begin{pmatrix} p^{\frac{k-1}{2}} & 0 \\ 0 & p^{\frac{k-1}{2}} \end{pmatrix}$ $\left(\text{ie} \begin{pmatrix} |\,|^{\frac{k-1}{2}} & 0 \\ 0 & |\,|^{\frac{k-1}{2}} \end{pmatrix} \right)$

(example where $WD(\ )^{ss} \not\equiv$ is needed)

• ($i_1 = 1-k$, [illegible])

Then $\rho \otimes \pi$ has an invariant norm, but for (φ, D) one has to take ↳ (because at least for $k < 2p$)

$D := L_0 e_1 \oplus \dots \oplus L_0 e_n$, $N : D \to D$ linear

$\varphi : D \to D$ bij. semi-linear, $N\varphi = p\varphi N$

$(\mathrm{Fil}^i D_{L'})_{i \in \mathbb{Z}}$ = decreasing exhaustive separated filtration on $D_{L'} := L' \otimes D$

$$t_N(D) = t_N\left(\Lambda^n_{L_0} D\right) = v_p\left(\frac{\varphi(e_1 \wedge \dots \wedge e_n)}{e_1 \wedge \dots \wedge e_n}\right)$$

$$t_H(D_{L'}) = t_H\left(\Lambda^n_{L'} D_{L'}\right) = \sum_{i \in \mathbb{Z}} i \dim_{L'} \frac{\mathrm{Fil}^i D_{L'}}{\mathrm{Fil}^{i+1} D_{L'}}$$

__Def__ (Fontaine): The filtration is (weakly) admissible if $t_N(D) = t_H(D_{L'})$ and for any $D' \subseteq D$ preserved by φ, N with the induced filtration on $D'_{L'}$, one has $t_H(D'_{L'}) \leq t_N(D')$.

"Hodge polygon under Newton polygon"

$$\begin{cases} \varphi(e_1) = p^{-\frac{k-1}{2}} e_1 \\ \varphi(e_2) = p^{-\frac{k-1}{2}} (e_1+e_2) \end{cases}$$ then there is a w. a. filtration given by: $\mathrm{Fil}^{-(k-1)+1} = \cdots = \mathrm{Fil}^0 = K\cdot(e_1+e_2)$.

(so φ is NOT semi-simple)

Why is this conjecture a first step towards p-adic Langlands (for de Rham Galois representation)? Because then, one might hope that a given specific weakly admissible filtration might "correspond" to a given specific norm on $\rho\otimes\pi$ (hence giving rise to a Banach space). And indeed, we will see later that, at least for $GL_2(\mathbb{Q}_p)$ and irreducible de Rham representation, such a phenomenon really happens.

Before going to special cases (for $GL_2(\mathbb{Q}_p)$), I would like to survey now some special or partial cases of the conjecture but still for $GL_{d+1}(L)$.

<u>Some cases</u>:

<u>prop.</u>: The central character of $\rho\otimes_K\pi$ is integral iff for any filtration satisfying $(*)$, one has $t_H(D_{L'}) = t_N(D)$.

<u>proof</u>: The central character of $\rho\otimes_K\pi$ is integral iff:

$$\mathrm{val}_L\big(\text{central char. of } \rho\,(\pi_L)\big) + \mathrm{val}_L\big(\text{central char of } \pi\,(\pi_L)\big) = 0.$$

One computes: (recall $\mathrm{val}_L(\pi_L)=1$)

$$\mathrm{val}_L\big(\text{c. ch. } \rho\,(\pi_L)\big) = -\sum_{j=1}^{d+1}\sum_\sigma \big(i_{d+2-j,\sigma} + (j-1)\big)$$

$$\mathrm{val}_L\big(\text{c. ch. } \pi\,(\pi_L)\big) = -\mathrm{val}_L\big((\det_K(r))(\text{arith. Frob. of } W(\overline{\mathbb{Q}}_p/L))\big)$$

$$+ [L:\mathbb{Q}_p]\frac{d(d+1)}{2}$$

Denoting $D_{\sigma'_0} = D \otimes_{L'_0 \otimes_{\mathbb{Q}_p} K, \sigma'_0 \otimes 1} K$ (where $\sigma'_0 : L'_0 \hookrightarrow K$),

one checks that $-\mathrm{val}_L\left(\left(\det_K(r)\right)(\text{arith. Frob})\right) = \frac{f}{f'} \mathrm{val}_L\left(\det_K\left(\varphi^{f'}|_{D_{\sigma'_0}}\right)\right)$

(note that $\varphi^{f'} : D_{\sigma'_0} \to D_{\sigma'_0}$ is K-linear). Hence, one has:

$$\begin{cases} \mathrm{val}_L\left(\text{c.ch. } \rho(\pi_L)\right) = -\left(\sum_{j=1}^{d+1}\sum_\sigma i_{j,\sigma}\right) - [L:\mathbb{Q}_p]\dfrac{d(d+1)}{2} \\ \mathrm{val}_L\left(\text{c.ch. } \pi(\pi_L)\right) = \dfrac{f}{f'} \mathrm{val}_L\left(\det_K\left(\varphi^{f'}|_{D_{\sigma'_0}}\right)\right) + [L:\mathbb{Q}_p]\dfrac{d(d+1)}{2} . \end{cases}$$

Now, one has:

$$t_H(D_{L'}) = \sum_\sigma \sum_{j=1}^{d+1} [K:L]\, i_{j,\sigma} \quad \text{and} \quad t_N(D) = [K:L]\frac{f}{f'} \mathrm{val}_L\left(\det_K\left(\varphi^{f'}|_{D_{\sigma'_0}}\right)\right)$$

hence $\mathrm{val}_L\left(\text{c.ch. } \rho(\pi_L)\right) + \mathrm{val}_L\left(\text{c.ch. } \pi(\pi_L)\right) = \dfrac{1}{[K:L]}\left(-t_H(D_{L'}) + t_N(D)\right)$. $\square$

<u>Corollary</u>: The conjecture holds if r is abs. irreducible (equiv. if π is supercuspidal).

<u>proof</u>:

- One can always write $\pi = \text{c-ind}_{UZ}^G \sigma$ where $Z = L^\times$ and U = some open compact open in G. Hence $\rho \otimes_K \pi = \text{c-ind}_{UZ}^G (\rho \otimes_K \sigma)$. We see that π has an invariant lattice iff $\rho \otimes_K \sigma$ has iff the central char. of $\rho \otimes_K \sigma$ is integral = central char. of $\rho \otimes_K \pi$.
- As the object $(\varphi, N, \mathrm{Gal}(L'/L), D) \in \mathrm{MOD}_{L'/L}$ corresponding to (r, N, V) by the previous equivalence of categories is irreducible, its only subobjects are 0 or itself. Hence, the weak admissibility conditions are just $t_H(D_{L'}) = t_N(D)$. The corollary therefore follows from the proposition.

In the same way, one can prove that if r is abs. indecomposable (equiv. π is a generalized Steinberg), then a filtration as in the conj. is (weakly) admissible iff $t_H(D_{L'}) = t_N(D)$. The following conjecture is thus a special case of the previous one:

Conj.: If π is a generalized Steinberg, then $\rho \otimes_k \pi$ admits an invariant norm iff its central character does.

Example 3: • $L = L' = \mathbb{Q}_p$, $d = 1$ and r is given by $\begin{pmatrix} |\,|^{k/2} & * \\ 0 & |\,|^{\frac{k-2}{2}} \end{pmatrix}$
• $i_1 = 1-k$, $i_2 = 0$

Then $\rho \otimes_k \pi = \mathrm{Sym}^{k-2} K^2 \otimes_k \text{Steinberg} \otimes |\det|^{\frac{k-2}{2}}$
where $\text{Steinberg} = \mathrm{Ind}_{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)}^{GL_2(\mathbb{Q}_p)} 1 \Big/ 1$. Teitelbaum and GK have proven that $\rho \otimes_k \pi$ has an invariant norm.

Thm (Schneider, Teitelbaum, B.): Assume that (r, N, V) is a direct sum of unramified characters, then (i) $\Rightarrow$ (ii) in the conjecture.

Sketch of proof: r: arith. Frob. of $W(\overline{\mathbb{Q}}_p/L) \longmapsto \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_{d+1} \end{pmatrix}$ $\quad z_i \in (K^\times)^{d+1}$

Let $U := GL_{d+1}(\mathcal{O}_L)$ and $G := GL_{d+1}(L)$. Let:

$$\mathcal{H}(G, 1_U) := \mathrm{End}_G\left(c\text{-}\mathrm{ind}_U^G 1_U\right) \simeq \{f : U\backslash G/U \xrightarrow[\text{c. support}]{} K\}$$

$$\mathcal{H}(G, \rho|_U) := \mathrm{End}_G\left(c\text{-}\mathrm{ind}_U^G \rho|_U\right) \simeq \{f : G \longrightarrow \mathrm{End}_K(V_\rho) \mid f(u_1 g u_2) = \rho(u_1) f(g) \rho(u_2)\} \text{ + cpct support}$$

then $i : \mathcal{H}(G, 1_U) \xrightarrow{\sim} \mathcal{H}(G, \rho|_U)$

$$f \longmapsto \left(g \mapsto f(g)\rho(g)\right).$$

let $T \subset G$ be the split torus and $T^\circ = T \cap U$, let:

$$\hat{\xi} : T/T^\circ \longrightarrow K, \quad \hat{\xi} = \mathrm{unr}(\xi_1) \otimes \mathrm{unr}(\xi_2)|\;|_L \otimes \cdots \otimes \mathrm{unr}(\xi_{d+1})|\;|_L^d$$

then it is a result of Dat that $\pi \simeq K \underset{\hat{\xi} \nearrow \mathcal{H}(G,1_U)}{\otimes} c\text{-}\mathrm{ind}_U^G 1_U$

where $\hat{\xi} : \mathcal{H}(G,1_U) \xhookrightarrow[\text{Satake map}]{} K[T/T^\circ] \xrightarrow{\hat{\xi}} K$ (remember π is L.l. modified).

Denote by $\rho^\circ|_U$ a U-lattice in ρ, then one has an associated norm on ρ, hence on $c\text{-}\mathrm{ind}_U^G \rho$, hence on $\mathrm{End}_G(c\text{-}\mathrm{ind}_U^G \rho)$, hence on $\mathcal{H}(G,\rho|_U)$. Denote by $\mathcal{B}(G,\rho|_U)$ the completion of $\mathcal{H}(G,\rho|_U)$ with respect to this norm. Then it can be shown that a K-point:

$\hat{\xi} : \mathcal{H}(G,\rho|_U) \xrightarrow[i^{-1}]{} \mathcal{H}(G,1_U) \xrightarrow{\text{as above}} K$ factors through $\mathcal{B}(G,\rho|_U)$ (i.e. that the K-point sends the unit ball of $\mathcal{H}(G,\rho|_U)$ to $\frac{1}{p^N}\mathcal{O}_K$ for $N \gg 0$)

iff it satisfies the inequalities:

$$\underset{\text{tuple}}{\overset{=}{d+1-}} \left\{ \left(\overbrace{\mathrm{val}_L \circ \hat{\xi}}^{\mathrm{val}_L(\xi_1),\, \mathrm{val}_L(\xi_2 \frac{1}{q}),\, \dots,\, \mathrm{val}_L(\xi_{d+1}\frac{1}{q^d})} + [L:\mathbb{Q}_p]\left(-\tfrac{d}{2}, -\tfrac{d}{2}+1, \dots, \tfrac{d}{2}\right)\right)^{\mathrm{dom}} \leqslant \left(\sum_\sigma a_{1,\sigma}, \dots, \sum_\sigma a_{d+1,\sigma}\right) + [L:\mathbb{Q}_p]\left(-\tfrac{d}{2}, -\tfrac{d}{2}+1, \dots, \tfrac{d}{2}\right) \right.$$

where $a_{j,\sigma} = -i_{d+2-j,\sigma} - (j-1)$

or equivalently the inequalities:

$$\left(\mathrm{val}_L(\xi_1), \dots, \mathrm{val}_L(\xi_{d+1})\right)^{\mathrm{dom}} \leqslant \left(\sum_\sigma a_{1,\sigma}, \dots, \sum_\sigma a_{d+1,\sigma}\right) + [L:\mathbb{Q}_p](0,1,\dots,d).$$

Now assume $\rho \otimes_K \pi$ has an invariant norm, then so does

$\underbrace{K \underset{\hat{\xi} \nearrow \mathcal{H}(G,\rho|_U)}{\otimes} c\text{-}\mathrm{ind}_U^G \rho|_U}$ which implies that the image of the

unit ball of $c\text{-}\mathrm{ind}_U^G \rho|_U$ in remains a lattice, which is easily seen to imply that $\hat{\xi} : \mathcal{H}(G,\rho|_U) \to K$ extends to $\mathcal{B}(G,\rho|_U)$ (i.e. has bounded values when restricted to $\mathcal{H}(G,\rho|_U)^\circ$), hence satisfies the above inequalities. Together with Prop. $\overset{\text{easy}}{\Longleftrightarrow}$ $\exists$ a weakly admissible filtration. $\square$