Jeudi 20/04/06

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Aspects of the p-adic local Langlands programme

Aspect IV : Modulo p.

For the group $GL_2(\mathbb{Q}_p)$, there is a nice mod. p local Langlands correspondence (which I am going to recall). This is closely related to the weight recipee in Serre's conjecture. Recently, Serre's conjecture has been generalized by Buzzard, Diamond, Jarvis and proved (?) by Gee. To turn this into a (semi-simple) mod. p correspondence, the first step is to construct new representations on the GL_2-side. One new representation was found (using global techniques) recently by Buzzard, Diamond and Emerton using results of Gee and Paskunas. I would like to explain here how one can use techni-ques introduced by V. Paskunas to construct locally new representations on the GL_2-side (work in progress with V. Paskunas).

The $GL_2(\mathbb{Q}_p)$-case. $B := B(\mathbb{Q}_p)$ = upper Borel

For simplicity, I use the notations $G := GL_2(\mathbb{Q}_p)$, $K := GL_2(\mathbb{Z}_p)$, $Z := \mathbb{Q}_p^\times$.

Let me start with the classification of smooth irreducible representations of $GL_2(\mathbb{Q}_p)$ over $\overline{\mathbb{F}}_p$ that admit a central character (a result essentially due to Barthel and Livné, with one case due to I):

- the 1-dim! representations (the characters)
- the principal series $\mathrm{Ind}_B^G \chi_1 \otimes \chi_2$ where $\chi_i : \mathbb{Q}_p^\times \to \overline{\mathbb{F}}_p^\times$ with $\chi_1 \neq \chi_2$
- the special serie Steinberg $\otimes\, \chi$ where St (= Steinberg) $= \dfrac{\mathrm{Ind}_B^G 1}{1}$
- the supersingular representations $\dfrac{c\text{-}\mathrm{ind}_{KZ}^G \mathrm{Sym}^r \overline{\mathbb{F}}_p^2}{(T)} \otimes \chi$ where $0 \leq r \leq p-1$.

I have to say a word for the last case: $\mathrm{Sym}^r \overline{\mathbb{F}}_p^2$ is a represent. of KZ via $z \mapsto 1$ and $K \twoheadrightarrow GL_2(\mathbb{F}_p)$, $\text{c-ind}_{KZ}^G \mathrm{Sym}^r \overline{\mathbb{F}}_p^2$ means functions $f : G \longrightarrow \mathrm{Sym}^r \overline{\mathbb{F}}_p^2$ with compact support modulo Z s.t. $f(kzg) = k f(g)$.

One has $\mathrm{End}_G\left(\text{c-ind}_{KZ}^G \mathrm{Sym}^r \overline{\mathbb{F}}_p^2\right) = \overline{\mathbb{F}}_p[T]$ where T can be described as follows: $\exists$ a unique function $\varphi : G \longrightarrow \mathrm{End}_{\overline{\mathbb{F}}_p}(\mathrm{Sym}^r \overline{\mathbb{F}}_p^2)$ with support on $KZ\begin{pmatrix} 1 & 0 \\ 0 & 1/p \end{pmatrix}KZ$ s.t. $\varphi\left(k_1 \begin{pmatrix} 1 & 0 \\ 0 & 1/p \end{pmatrix} k_2\right) = k_1 \circ \begin{array}{c} \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & 1 \end{pmatrix} \\ x^r \cdots y^r \end{array} \circ k_2$, then one has:

$$T([g,v]) = \sum_{g'KZ \in G/KZ} [gg', \varphi(g'^{-1})(v)]$$

where $[g,v] \in \text{c-ind}_{KZ}^G \mathrm{Sym}^r$ is defined by: $\begin{cases} [g,v](g') = g'g \cdot v & \text{if } g'g \in KZ \\ [g,v](g') = 0 & \text{if } g'g \notin KZ. \end{cases}$

Moreover, among the supersingular, one has the intertwining:

$$\frac{\text{c-ind}_{KZ}^G \mathrm{Sym}^r \overline{\mathbb{F}}_p^2}{T} \simeq \frac{\text{c-ind}_{KZ}^G \mathrm{Sym}^r \overline{\mathbb{F}}_p^2}{T} \otimes \mathrm{unr}(-1)$$

$$\frac{\text{c-ind}_{KZ}^G \mathrm{Sym}^r \overline{\mathbb{F}}_p^2}{T} \simeq \frac{\text{c-ind}\, \mathrm{Sym}^{p-1-r} \overline{\mathbb{F}}_p^2}{T} \otimes (\omega^r \circ \det)$$

Now, the semi-simple mod p local Langlands correspondence for $GL_2(\mathbb{Q}_p)$ can be stated as follows:

$$\left(\mathrm{Ind}_B^G \chi_2 \otimes \chi_1 \omega^{-1}\right)^{ss} \oplus \left(\mathrm{Ind}_B^G \chi_1 \otimes \omega^{-1}\chi_2\right)^{ss} \longleftrightarrow \bar{\rho} \simeq \begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix}$$

ω_2 = fund. char. of level 2

$$\frac{\text{c-ind}_{KZ}^G \mathrm{Sym}^r \overline{\mathbb{F}}_p^2}{(T)} \longleftrightarrow \bar{\rho}|_{\mathrm{Inert.}} \simeq \begin{pmatrix} \omega_2^{r+1} & 0 \\ 0 & \omega_2^{p(r+1)} \end{pmatrix}.$$

centr. char $\simeq \omega^{r+1}$

Rk: Emerton has given a non semi-simple refinement of this correspondence.

Relation with Serre's conjecture (the weight part): $\quad 0 \leq r \leq p-1$

2 "companion" weights: Sym^r, $\mathrm{Sym}^{[p-3-r]} \otimes \omega^{r+1} \circ \det \longleftrightarrow \bar\rho|_I \simeq \begin{pmatrix} \omega^{r+1} & 0 \\ 0 & 1 \end{pmatrix}$

2 "intertwined" weights: Sym^r, $\mathrm{Sym}^{p-1-r} \otimes \omega^r \circ \det \longleftrightarrow \bar\rho|_I \simeq \begin{pmatrix} \omega_2^{r+1} & 0 \\ 0 & \omega_2^{p(r+1)} \end{pmatrix}$

rq: $\dfrac{\text{c-ind}\,\mathrm{Sym}^r}{T-\lambda} \simeq \mathrm{Ind}_B^G \mathrm{unr}(\lambda) \otimes \omega^r \mathrm{unr}(\lambda^{-1})$; $\dfrac{\text{c-ind}\,\mathrm{Sym}^{[p-3-r]}}{T-\lambda} \otimes \omega^{r+1} \circ \det \simeq \mathrm{Ind}_B^G \omega^{r+1} \mathrm{unr}(\lambda^{-1}) \otimes \omega\, \mathrm{unr}(\lambda)$

The $GL_2(\mathbb{Q}_{p^2})$-case: (generic case) Let me recall the list of Diamond weights in that case. I fix an embedding $\mathbb{F}_{p^2}^\times \hookrightarrow \overline{\mathbb{F}}_p^\times$ which, by LCFT, amounts to fixing $\omega_2 : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^2})^{ab} \longrightarrow \overline{\mathbb{F}}_p^\times$.

(r_0, r_1), $(r_0+1, p-2-r_1) \otimes \omega_2^{p-1+pr_2}$, $(p-2-r_0, r_1+1) \otimes \omega_2^{r_0+p(p-1)}$, $(p-3-r_0, p-3-r_1) \otimes \omega_2^{r_0+1+p(r_1+1)}$ $\longleftrightarrow \bar\rho|_I \simeq \begin{pmatrix} \omega_2^{r_0+1+p(r_1+1)} & 0 \\ 0 & 1 \end{pmatrix}$

(r_0, r_1), $(r_0-1, p-2-r_1) \otimes \omega_2^{p(r_1+1)}$, $(p-1-r_0, p-3-r_1) \otimes \omega_2^{r_0+p(r_1+1)}$, $(p-2-r_0, r_1+1) \otimes \omega_2^{r_0+1-p}$ $\longleftrightarrow \bar\rho|_I \simeq \begin{pmatrix} \omega_4^{r_0+1+p(r_1+1)} & 0 \\ 0 & \text{conj.} \end{pmatrix}$

where (s_0, s_1) stands for $\mathrm{Sym}^{s_0} \overline{\mathbb{F}}_p^2 \otimes \left(\mathrm{Sym}^{s_1} \overline{\mathbb{F}}_p^2\right)^{\mathrm{Frob}}$.
$\circlearrowleft$
$GL_2(\mathbb{F}_{p^2})$
(via chosen embedding)

We would like to build out of this a "semi-simple" mod p correspondence, or at least to associate to $\bar\rho$ a semi-simple $\pi(\bar\rho)$. In the sequel, I focus on the irreducible case. If the situation was truly the same as for $GL_2(\mathbb{Q}_p)$, we would have intertwinings $\dfrac{c\text{-ind}(r_0,r_1)}{(T)} \simeq \dfrac{c\text{-ind}(r_0-1,p-2-r_1)}{T} \otimes (\omega_2^{p(r_1+1)} \circ \det) \simeq \dfrac{c\text{-ind other weights}}{T}$ (4)

and these representations would be irreducible. However, neither is true. Instead, one has:

<u>Thm (Paskunas)</u>: (i) One has embeddings:

$$\begin{matrix} c\text{-ind}_{KZ}^{G}(s_0-1,p-2-s_1)\otimes\omega_2^{p(s_1+1)} \\ \oplus \\ c\text{-ind}_{KZ}^{G}(p-2-s_0,s_1-1)\otimes\omega_2^{s_0+1} \end{matrix} \hookrightarrow \frac{c\text{-ind}(s_0,s_1)}{(T)}$$

(forgetting the JH factors with "negative" weights)

(ii) Denote by $Q(s_0,s_1)$ the cokernel of the above embedding, then $Q(s_0,s_1) \simeq Q(p-1-s_0,p-1-s_1)\otimes\omega_2^{s_0+ps_1}$.

Using the above theorem together with results of T. Gee, the following theorem was proven during the Palo-Alto conference:

<u>Thm</u> (Buzzard, Diamond, Emerton, Gee, ...).

The 4 above representations admit at least one common irreducible admissible quotient.

This quotient is built on the cohomology of a Shimura curve. The main step is as follows: take the weight (r_0,r_1) and a global $\bar\rho$ which is irreducible as above when restricted to I, then one has a map:

$$\frac{c\text{-ind}(r_0,r_1)}{T} \longrightarrow \bar\rho \text{ part of cohomology}.$$

Now, by Paskunas' theorem, as the weight $(p-1-r_0,p-1-r_1)\otimes\omega_2^{r_0+pr_1}$ is NOT in the list of weights, necessarily (using Gee) the composed map $c\text{-ind}(r_0-1,p-2-r_1)\otimes\omega_2^{p(r_1+1)} \longrightarrow \dfrac{c\text{-ind}(r_0,r_1)}{T} \longrightarrow \bar\rho$ part is non-zero and factors through $\dfrac{\text{c-ind}(r_0-1, p-2-r_1)}{T} \otimes \omega_2^{p(r_1+1)}$. Repeating this game again, one finds each of the 4 weights appearing. $\square$ Then the image of $\dfrac{\text{c-ind}()}{T}$ is an irreducible supersingular.

I would like to explain now a local construction of such common quotients based on techniques introduced by Paskunas:

<u>A local construction of common quotients.</u>

let $\sigma_1 := (r_0, r_1)$, $\sigma_2 := (r_0-1, p-2-r_1) \otimes \omega_2^{p(r_1+1)}$, $\sigma_3 := (p-1-r_0, p-3-r_1) \otimes \omega_2^{r_0+p(r_1+1)}$

$\sigma_4 := (p-2-r_0, r_1+1) \otimes \omega_2^{r_0+p(p-1)}$.

For σ a weight, denote by $\mathrm{Inj}\,\sigma$ the injective envelope of σ in the category of smooth representations of K over $\overline{\mathbb{F}}_p$. It is of infinite dimension and is unique up to non-unique isomorphism, and its K-socle is σ.

Making p act trivially, let V be the following representation of KZ:

$$V := \mathrm{Inj}\,\sigma_1 \oplus \mathrm{Inj}\,\sigma_2 \oplus \mathrm{Inj}\,\sigma_3 \oplus \mathrm{Inj}\,\sigma_4 .$$

let χ_i be the character by which I acts on $\sigma_i^{I(1)} = \overline{\mathbb{F}}_p v_i$ where

$$\underset{\text{pro-}p\text{ Iwahori}}{\underset{\parallel}{I(1)}} = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in K,\ a \equiv 1(p),\ d \equiv 1(p),\ c \equiv 0(p)\right\} \text{ and } \underset{\text{Iwahori}}{\underset{\parallel}{I}} = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in K,\ c \equiv 0(p)\right\}.$$

One has the following lemma of group theory:

<u>Lemma 1:</u> There is a (non-canonical) I-equivariant isomorphism:

$$\mathrm{Inj}\,\sigma\big|_I \simeq \bigoplus_{\substack{\chi \text{ char of } I \text{ s.t.} \\ \sigma \text{ is a subquotient} \\ \text{of } \mathrm{Ind}_{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)}^{GL_2(\mathbb{F}_q)} \chi}} \underbrace{\mathrm{Inj}\,\chi}_{\text{= injective envelope in the category of smooth representations of } I \,/\, \overline{\mathbb{F}}_p} .$$

If χ is a character of I, say $\chi\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \chi_1(a)\chi_2(d)$, let χ^s be the character $\chi^s\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \chi_2(a)\chi_1(d)$. Using lemma 1, one can see that:

$$\underset{\chi_1}{\mathrm{Inj}\,\sigma_1}\Big|_I \simeq \mathrm{Inj}\,\chi_1 \oplus \mathrm{Inj}\,\chi_4^s \oplus \mathrm{Inj}\,\chi_4 \oplus \mathrm{Inj}\,\chi_1^s$$

$$\underset{\chi_2}{\mathrm{Inj}\,\sigma_2}\Big|_I \simeq \mathrm{Inj}\,\chi_2 \oplus \mathrm{Inj}\,\chi_1^s \oplus \mathrm{Inj}\,\chi_1 \oplus \mathrm{Inj}\,\chi_2^s$$

$$\underset{\chi_3}{\mathrm{Inj}\,\sigma_3}\Big|_I \simeq \mathrm{Inj}\,\chi_3 \oplus \mathrm{Inj}\,\chi_2^s \oplus \mathrm{Inj}\,\chi_2 \oplus \mathrm{Inj}\,\chi_3^s$$

$$\underset{\chi_4}{\mathrm{Inj}\,\sigma_4}\Big|_I \simeq \mathrm{Inj}\,\chi_4 \oplus \mathrm{Inj}\,\chi_3^s \oplus \mathrm{Inj}\,\chi_3 \oplus \mathrm{Inj}\,\chi_4^s.$$

Now, we can make a partition of these injective envelopes as follows (see above)

<u>Lemma 2</u> (Paskunas): Let N be the normalizer of I inside G. Then one can extend $\overset{\text{non-uniquely}}{\vee}$ the action of I on $\mathrm{Inj}\,\chi \oplus \mathrm{Inj}\,\chi^s$ to an action of N such that $p \in Z$ act trivially and $\begin{pmatrix} 0 & 1 \\ p & 0\end{pmatrix}$ exchanges $\mathrm{Inj}\,\chi$ and $\mathrm{Inj}\,\chi^s$.

<u>proof</u>: Let $\Pi := \begin{pmatrix} 0 & 1 \\ p & 0\end{pmatrix}$ and $(\mathrm{Inj}\,\chi)^{\Pi}$ the representation $\mathrm{Inj}\,\chi$ but with the I-action twisted by $g \ast v := \Pi g \Pi^{-1} v$. Then, as $\chi^{\Pi} \simeq \chi^s$, we have $0 \to \chi^s \hookrightarrow (\mathrm{Inj}\,\chi)^{\Pi}$. Using the fact that $\mathrm{Hom}_I(V, (\mathrm{Inj}\,\chi)^{\Pi}) = \mathrm{Hom}_I(V^{\Pi}, \mathrm{Inj}\,\chi)$ and that $\mathrm{Inj}\,\chi$ is injective, one easily gets that $(\mathrm{Inj}\,\chi)^{\Pi}$ is also injective. By unicity of the injective envelope of χ^s, there is an I-equivariant isomorphism $\phi : (\mathrm{Inj}\,\chi)^{\Pi} \xrightarrow{\sim} \mathrm{Inj}\,\chi^s$ and we define the action of Π (non-canonical) as: $\Pi(v_\chi \oplus v_{\chi^s}) := \phi^{-1}(v_{\chi^s}) \oplus \phi(v_\chi)$. $\square$

- Let W be the following representation of N:

$$W := \bigoplus_{\substack{\text{pairs } \chi, \chi^s \\ \text{above}}} \left(\operatorname{Inj}\chi \oplus \operatorname{Inj}\chi^s \right) \quad \text{(the action of } N \text{ here is non-canonical)}$$

Fix an ZI-equivariant isomorphism $\phi : V \xrightarrow{\sim} W$.

<u>Lemma 3</u> (Paskunas): There is a unique smooth G-representation Ω such that $\Omega|_{KZ} \xrightarrow{\sim} V$, $\Omega|_N \xrightarrow{\sim} W$ and the diagram:

$$\begin{tikzcd} \Omega \arrow[r, "\sim"] \arrow[d, equal, "\mathrm{Id}"'] & V \arrow[d, "\phi", "\wr"'] \\ \Omega \arrow[r, "\sim"] & W \end{tikzcd}$$

commutes.

Unicity is coming from the Iwahori decomposition $G = I \langle s, \Pi \rangle I$ where $s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\Pi = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$. The proof of existence uses coefficient systems on the tree of PGL_2.

One has $\operatorname{soc}(\Omega|_K) = \operatorname{soc}(V|_K) = \sigma_1 \oplus \sigma_2 \oplus \sigma_3 \oplus \sigma_4$ (as $\operatorname{soc}(\operatorname{Inj}\sigma_i) = \sigma_i$). Let π be the G-subrepresentation of Ω generated by $\operatorname{soc}(\Omega|_K)$.

<u>Thm</u>: π is irreducible admissible supersingular and is a common quotient of $\frac{\text{c-ind}\,\sigma_i}{T}$.

<u>proof</u>: • admissibility: one has $\pi^{I(1)} \subset \Omega^{I(1)} = \bigoplus_{(\chi,\chi^s)} (\operatorname{Inj}\chi)^{I(1)} \oplus (\operatorname{Inj}\chi^s)^{I(1)}$

using the general lemma that $\qquad = \bigoplus_{(\chi,\chi^s)} \chi \oplus \chi^s$

if H is a profinite group and $H(1) \subseteq H$ a normal pro-p-group, then

$$(\mathrm{Inj}\,\rho)^{H(1)} = \text{injective envelope of } \rho \text{ in the category of}$$

f.d. $\overline{\mathbb{F}}_p$-repres. of $H/H(1)$ (here, ρ = irred. rep. of H over $\overline{\mathbb{F}}_p$, hence with $H(1)$ acting trivially). When $\# H/H(1)$ is prime to p, just get ρ.

So $\pi^{I(1)}$ is f.d. $\Rightarrow$ π is admissible.

- irreducibility: let $\pi' \subseteq \pi$ be a non-zero G-subspace. Then

$$0 \neq \mathrm{Soc}(\pi'|_K) \subseteq \mathrm{Soc}(\pi|_K) \subseteq \underbrace{\sigma_1 \oplus \sigma_2 \oplus \sigma_3 \oplus \sigma_4}_{\text{all distinct}}.$$ Say σ_3 is in

$\pi'|_K$, then $\Pi(\sigma_3^{I(1)}) \in \pi'$, but $\Pi(\sigma_3^{I(1)}) \in \mathrm{Inj}\,\sigma_4$ (see the previous pairings), hence σ_4 is contained in the K-sub-representation of $\mathrm{Inj}\,\sigma_4$ generated by $\Pi(\sigma_3^{I(1)})$ (as $\mathrm{Inj}\,\sigma_4$ is an essential extension of σ_4), hence $\sigma_4 \subseteq \pi'|_K$ and we can look at $\Pi(\sigma_4^{I(1)})$ and play the same game again. We get that all σ_i are in $\pi'|_K \Rightarrow \pi' = \pi$.

useless [• supersingular: let v_i be a $\overline{\mathbb{F}}_p$-basis of $\sigma_i^{I(1)}$ and $w_i := \Pi \cdot v_i$. Then $\overline{\mathbb{F}}_p v_i \oplus \overline{\mathbb{F}}_p w_i \subset \Omega^{I(1)}$ is stable under the action of $\mathrm{End}_G(\text{c-}\mathrm{Ind}_{I(1)}^G 1)$ and is a supersingular Hecke module in the sense of Vignéras (this is a computation).] (Also, $\pi^{I(1)}$ contains $\oplus\, \sigma_i^{I(1)}$, hence is of dim ≥ 2, hence π cannot be a principal series (because the dim would then be 2).)

The final part follows from Barthel and Livné. $\square$

However, there seems to be several such π, ie changing the action of Π in lemma 2

really leads to several supersingular, all being irreducible quotients of $\dfrac{c\text{-ind}\,\sigma_i}{(T)}$. For instance, let us consider the last step in the previous proof: $\Pi v_i = w_i$, $\Pi w_i = v_i$. Change the action of Π such that $\Pi v_i = \lambda_i w_i$, $\Pi w_i = \lambda_i^{-1} v_i$, then the representation π you get $\underset{\text{(to be confirmed)}}{\underline{\text{is different if } \prod \lambda_i \neq 1}}$ (and for each value of $\prod \lambda_i$, get a different π).
(if everything OK)

The theorem can be generalized to a more general situation (cycles of any cardinality etc.) but there are more and more pb. of choices. So the picture for mod. p local correspondence is still not clear.