Feb 16 2006. Thursday. 1:00 PM — 2:30 P.M. 5th lecture

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Goal of next lecture or 2
to show you what a spectral curve is.

Working definition of p-adic modular forms

Recall $X_0(1)$ = cpt R.S
$= \{\infty\} \cup Y_0(1)$

Points on $Y_0(1)$ $\longleftrightarrow$ Iso. classes of ell. curves / $\mathbb{C}$

$$Y_0(1) = {}_{SL_2(\mathbb{Z})}\backslash \mathfrak{h} \xrightarrow[\cong]{j} \mathbb{C}$$
$$\tau \longmapsto j(\tau) \in \mathbb{C}$$

If E is an ell. curve/$\mathbb{C}$ then it has a j-invariant $j(E) \in \mathbb{C}$
& Iso class of E is determined by $j(E)$

All works in alg geom.
All works over an arbitrary alg. closed field.
Let's think about the picture over $\mathbb{C}_p = \hat{\bar{\mathbb{Q}}}_p$

— First let's think ...

$X_0(1)/\overline{\mathbb{F}_p} \cong \mathbb{P}^1$

$\therefore$ pts are $\infty \cup \overline{\mathbb{F}_p}$

cusp.

<u>Fact</u> There are a non-zero finite number of "super singular j-invariants"

$(\approx p/12)$

& all rest is ordinary

$E/\overline{\mathbb{F}_p}$. $\quad \# E[p](\overline{\mathbb{F}_p}) = \begin{cases} 1 & \text{(s.s.)} \\ p & \text{ord.} \end{cases}$

<u>Picture</u>

$X_0(1)_{\overline{\mathbb{F}_p}}$

finite set of s.s pts.

$\mathbb{C}_p \supseteq \mathcal{O}_{\mathbb{C}_p} \supseteq m \quad \& \quad \mathcal{O}_{\mathbb{C}_p}/m \cong \overline{\mathbb{F}_p}$

<u>Picture over $\mathbb{C}_p$</u>

$\mathbb{P}^1 = X_0(1)_{\mathbb{C}_p}$

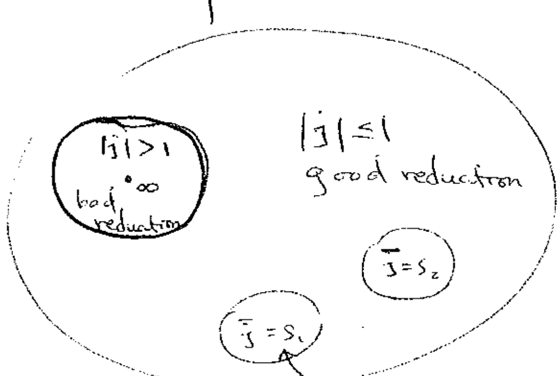
Idea: If $E/\mathbb{C}_p$ is an elliptic curve then E has either bad <u>reduction</u> ($\Leftrightarrow |j(E)| > 1$) non-integral

or it has good <u>reduction</u>.

$\overline{E}$ ($\Leftrightarrow |j(E)| \leq 1$)

& $j(\overline{E}) = \overline{j(E)}$

$X_0(1)/\mathbb{C}_p$



$j = \frac{1}{q} + \cdots$

open discs corresponding to elliptic curves with s.s. reduction

$\downarrow$

$X(1)$

Fact: There's a good theory of p-adic Riemann surfaces

& $X_0(1)^{an}$ is a good example of rigid-analytic projective line — but also the open discs & their complements

Define $X_0(1)^{ord} = X_0(1)^{an}$ minus supersingular discs

Fact: $X_0(1)^{ord}$ is a p-adic R.S. & one can ~~talk~~ about holomorphic / rigid-analytic ftns on $X_0(1)^{ord}$

Fresnel - Van der Put is a good place to look for this

One concrete example.

say only one s.s. disc

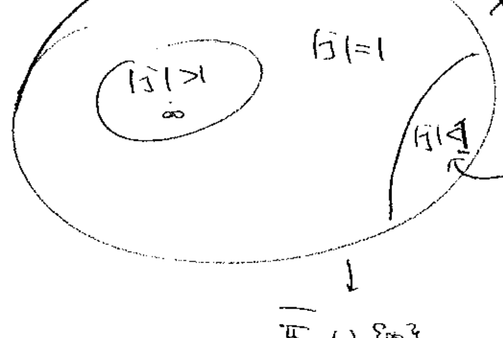
Remember this: $X_0(1)^{ord} \cong$ closed disc $(p = 2, 3, 5, 7, 13)$

Fact: If D = closed unit disc $\{|t| \leq 1\}$ over a complete field K ~~then~~ the holo. ftns on $D = K\langle T \rangle$

$$= \sum_{n \geq 0} a_n T^n \; : \; a_n \to 0 \text{ as } n \to +\infty$$

If $p = 2, 3, 5$
then unique ss j-invt is $j = 0$ in char p.

picture



$j = q^{-1} + 744 + \dots$

& So we could use

$$t = \frac{1}{j} = q - 744 q^2 + \dots \text{ to be our parameter.}$$

Conclusion : Space of p-adic modular functions

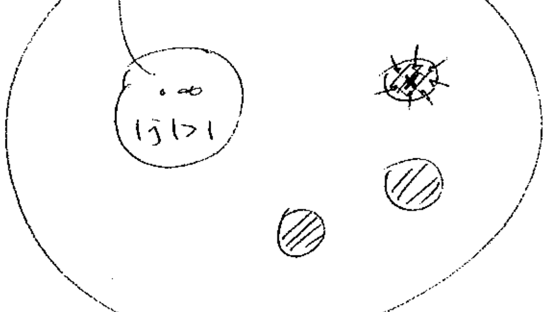
which is, by definition the holo. fns on $X_0(1)^{ord}$, is simply the space $\mathbb{C}_p\langle z\rangle$ for $p=2,3,5$.

Rmk : space of classical modular fons of wt 0 & level p is a global sections of $\omega^{\otimes 0}=\mathcal{O}$ on the cpt curve $X(p)$ $= \underline{\text{consts}}$.

Space of fns on $X_0(1)^{ord}$ = p-adic modular fns is clearly ∞-dimensional

$$X_0(1)^{ord} = X_0(1) \setminus \text{s.s. discs}$$

on this open disc, there's a holomorphic fns called q



$|q|<1$ on this disc.

If $E/\mathbb{C}_p$ has bad reduction, then $E=\mathbb{C}_p^\times/\langle q\rangle$. $q\in\mathbb{C}_p$ $|q|<1$
$q = q(E)$

Any fn on $X_0(1)^{ord}$ restricts to a fn on "bad reduction disc"

$\therefore$ is a power series in q.

$\sum a_n T^n$ converges $\forall T=t$, $|t|<1$

— this is the q-expansion of the p-adic modular fn.

Slight generalization

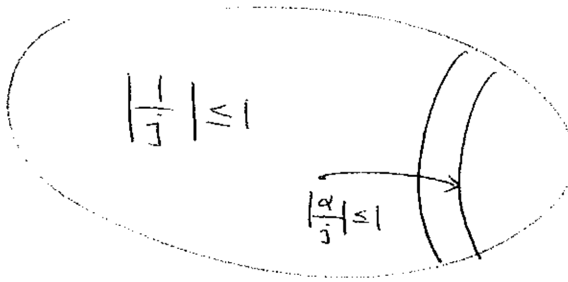
An overconvergent p-adic modular fn is a fn on $X_0(1)^{ord}$ that extends slightly into the supersingular discs

$\exists$ notion of "r-overconvergent p-adic modular ftns"

says exactly how far into s.s regions we can go

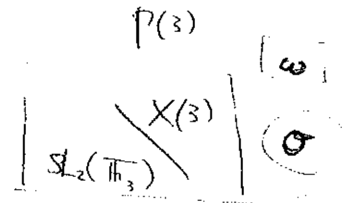
e.g. if $p = 2, 3, 5$, then $\mathbb{Q}_p\langle \frac{1}{j} \rangle$ is p-adic modular ftns

if $\alpha \in \mathbb{C}_p$ with $|\alpha| < 1$ then $\mathbb{Q}_p\langle \frac{\alpha}{j} \rangle$ = holo ftns on a slightly bigger disc



$X_0(1)$ no ω

I am going to explicitly compute one Hecke operator on 2-adic & overconverget 2-adic modular ftns.



Recall the Δ-ftn $\Delta(q) = q \prod_{n \ge 1} (1 - q^n)^{24}$. wt 12 level 1 mod. form

$\Delta(q^2)$ & $\Delta(q)$ are wt 12 level 2 modular forms

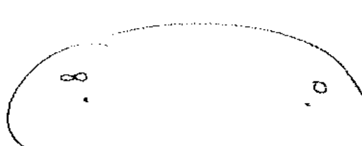
q-expansion principle $\quad > \boxed{\omega^{\otimes 12}}$ $\quad X_0(2)/\mathbb{Q}$

$\Rightarrow$ these ftns are algebro geometric & defined over $\mathbb{Q}$

Recall Δ is non-vanishing on $\mathfrak{h}$.

$\Delta(q^2)$ & $\Delta(q)$ are non-vanishing sections of $\omega^{\otimes 12}$ on $Y_0(2)/\mathbb{Q}$.

Picture$/\mathbb{C}$



$X_0(2) \cong \mathbb{P}^1$

$\pi^{-1}(\infty) = \infty + 0 + 0$

$\pi \downarrow 3$

$\infty \in X_0(1) \cong \mathbb{P}^1$

Δ on $X_0(1)$ has a simple zero @ ∞
& no other zeros

$\pi^*\Delta$ on $X_0(2)$ has a simple zero @ ∞
double zero @ 0

& $\Delta(q^2) = q^2 - 24 q^4 + \cdots$ has a double zero @ ∞.
& simple zero @ 0.

Set $f = \dfrac{\Delta(q^2)}{\Delta(q)}$

$= q\prod(1+q^n)^{24}$ (exercise)

$= q + 24q^2 + \cdots$

is non-vanishing on $Y_0(2)$ simple zero at ∞
simple pole at 0.

$f : X_0(2) \xrightarrow{\simeq} \mathbb{P}^1/\mathbb{C}$

$f : X_0(2) \xrightarrow{\simeq} \mathbb{P}^1/\mathbb{Q}$

Riemann-Hurwitz

$$\boxed{2(1-g) = 2n(1-g') - \deg R \qquad \deg R = \Sigma(e_i - 1) \qquad \underset{g}{C} \xrightarrow[\deg f = n]{f} \underset{g'}{C'}}$$

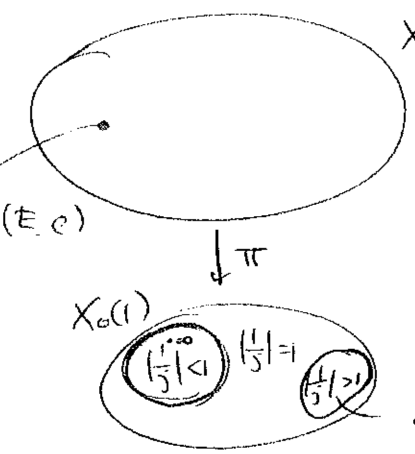
$$\begin{tikzcd}
X_0(2) \arrow[r, "f", "\cong"'] \arrow[d, "\text{projection}"] \arrow[dr] & \mathbb{P}^1 \arrow[d, dashed, "\deg 3"] \\
X_0(1) \arrow[r, "j", "\cong"'] & \mathbb{P}^1
\end{tikzcd}$$

& one checks that in fact

$$\frac{1}{j} = \frac{f}{(1+256f)^3}$$

$256 = 2^8$.

<u>$p = 2$</u>



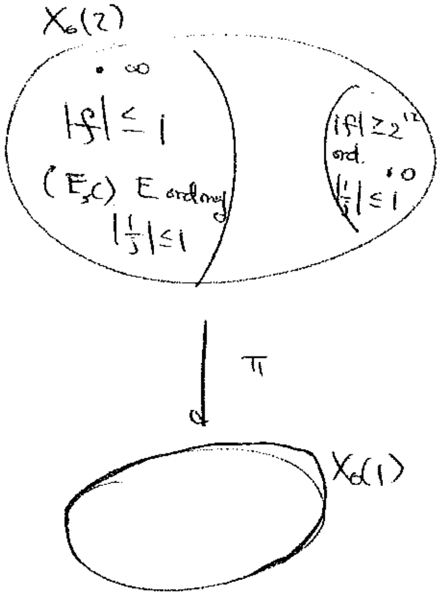
a point is a pair (E, C)
E : ell. curve$/\mathbb{Q}_2$
$C \subset E$ subgp order 2

s.s. region

What is the preimage of $|\frac{1}{j}| \le 1$ etc under π?

* If $|f| \le 1$,
then $|\frac{1}{j}| = |f| \le 1$
ordinary

$|f| \le 1$



$$|\tfrac{1}{j}|$$

$\ast$ If $|f| \ge 2^{12}$ $\qquad \frac{1}{j} = \frac{f}{(1+256f)^3}$

then $\left|(1+\underset{2^8}{\underset{\|}{256}}f)^3\right|_2 = |f|^3 \div 2^{24}$

$$\therefore \left|\frac{1}{j}\right| = \frac{|f| \cdot 2^{24}}{|f|^3} = \frac{2^{24}}{|f|^2} \le 1$$

If $|f| < 2^8$

then $|256 f| < 1$ $\therefore$ $|\text{denominator}| = 1$ & $\left|\frac{1}{j}\right| = |f|$

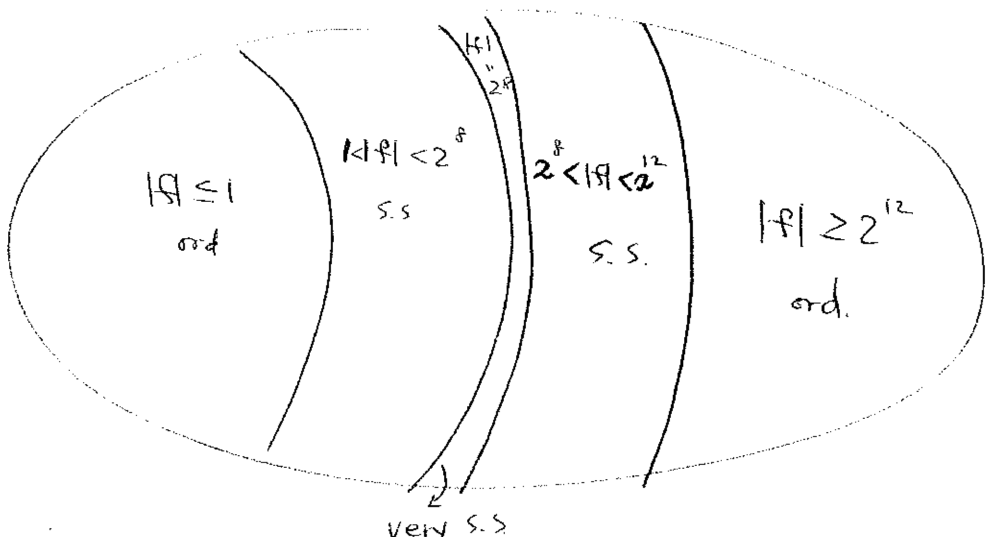
if $1 < |f| < 2^8$ then non-ordinary

If $|f| > 2^8$

then $|1 + 256 f| = |f| \div 2^8$ $\quad \therefore \left|\frac{1}{j}\right| = \frac{2^{24}}{|f|^2} < 2^8$

Finally if $|f| = 2^8$

then $|1 + 256 f| \le 1$ $\quad \therefore \left|\frac{1}{j}\right| \ge |f|$ $\quad \therefore \left|\frac{1}{j}\right| \ge 2^8$.



in terms of $\frac{1}{j}$.

$$\frac{1}{j} = \frac{f}{(1+256f)^3} \qquad \therefore \quad \frac{1}{j}(1+256f)^3 = f$$

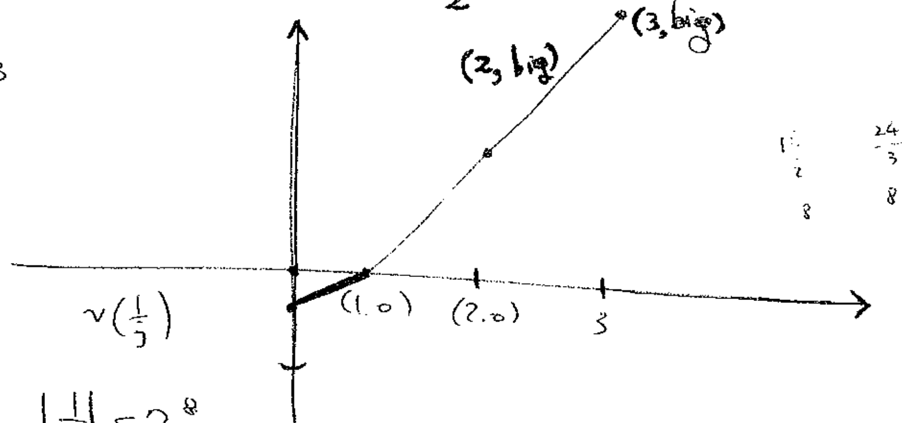
$\therefore$ f satisfies a cubic poly coeff involving $\frac{1}{j}$

Say $\left|\frac{1}{j}\right| < 2^8$.

Then $\frac{1}{j} + \left(3\times 256 \times \frac{1}{j} - 1\right)\cdot f + \left(\frac{1}{j}\right)\cdot 3\cdot 256^2 \cdot f^2 + \frac{1}{j} 256^3 \cdot f^3 = 0$

unit, 2^{16}, 2^{24}

NP is



Given $\frac{1}{j}$ with $\left|\frac{1}{j}\right| < 2^8$.

∴ there are 3 choices f. one of which has norm equal to $\left|\frac{1}{j}\right|$

Consequences:

Projection map gives an isomorphism $\{|f| \le 1 \text{ in } X_0(2)\}$

$$\cong$$

$\{\left|\frac{1}{j}\right| \le 1 \text{ in } X_0(1)\} = X_0(1)^{ord}$.

& Furthermore if $t < 2^8$

it even gives an isom $\{|f| \le t \text{ in } X_0(2)\}$

$$\cong$$

$\{\left|\frac{1}{j}\right| \le t \text{ in } X_0(1)\}$

2-adic modular forms $= \mathbb{Q}_2\langle f\rangle$

& if $\alpha \in \mathbb{Q}_2$, $2^{-8} < \alpha < 1$

then $\mathbb{Q}_2\langle \alpha\cdot f\rangle$ is a space of overconverged 2-adic forms.

$$\frac{1}{j} = \frac{f}{(1+256f)^3}$$

$\frac{1}{j} = f(1 - 768f + \cdots)$ ∴ f = power series in $\frac{1}{j}$

$= f\cdot\mathbb{Z}_2[[2^8 f]]$ converges if $\left|\frac{1}{j}\right| < 2^8$.

$$\therefore \frac{2^8}{j} \in 2^8 f\, \mathbb{Z}_2[[2^8 f]] = 2^8 f + \dots$$

$$\therefore 2^8 f \in \frac{2^8}{j}\, \mathbb{Z}_2[[\tfrac{2^8}{j}]]$$

Our conclusion is slightly **strange** :

if $E/\mathbb{Q}_2$ is an ell. curve & $|j(E)| > \frac{1}{2^8}$

there's a natural $f \in \mathbb{Q}_2$

i.e a natural pt on $X_0(2)$ lifting E

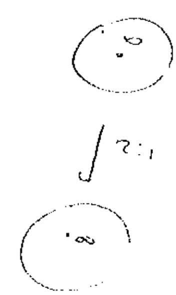
we have singled out a subgp of order 2

If $|j(E)| \geq 1$, then E either has bad or good ordinary

& we can guess the subgp

1) E has <u>good</u> <u>ord.</u> red. $\overline{E}$

$$\begin{array}{cl} E(\mathbb{Q}_2)[2] & \text{order } 4 \\ \downarrow & \\ \overline{E}(\overline{\mathbb{F}}_2)[2] & \text{order } 2 \end{array}$$



& the "canonical" subgp is <u>kernel</u> of <u>reduction</u>

2) E has bad reduction

$$E_2 = \mathbb{Q}_2^\times / \langle q \rangle$$
$$\cup\!\!|$$
$$\pm 1$$

If $\frac{1}{2^8} < |j(E)| < 1$

then E has good s.s. red

— so how to chose a canonical subgp?

<u>Recall</u> that in **char 2**

the s.s ell. curve is $y^2 + y = x^3 + x^2 \ / \overline{\mathbb{F}}_2$

So let's consider the ell. curve/$\mathbb{Q}_2$

$$y^2+y+axy=x^3+x^2, \quad a\in\mathbb{Q}_2,\ |a|<1$$

& assume $|a| = 1-\varepsilon$.

(not too super singular)

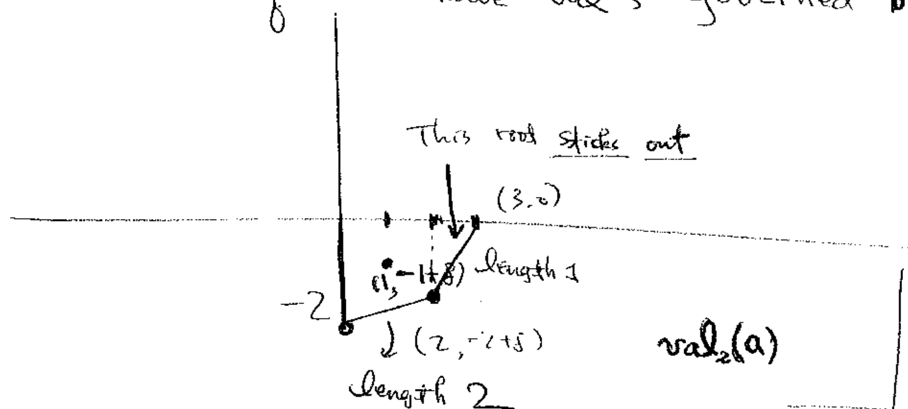
2-torsion: set

$$Y = y+\frac{1}{2}+\frac{ax}{2}$$

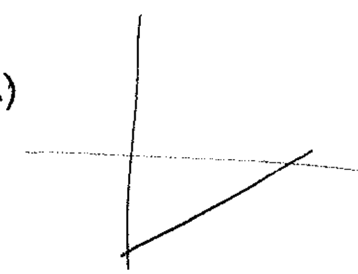
$\therefore Y^2$ = cubic in X.

$$= X^3+\left(\frac{a^2}{4}+1\right)x^2+\frac{a}{2}x+\frac{1}{4}.$$

3 roots of equation have val's governed by N.P.



$val_2(a)$



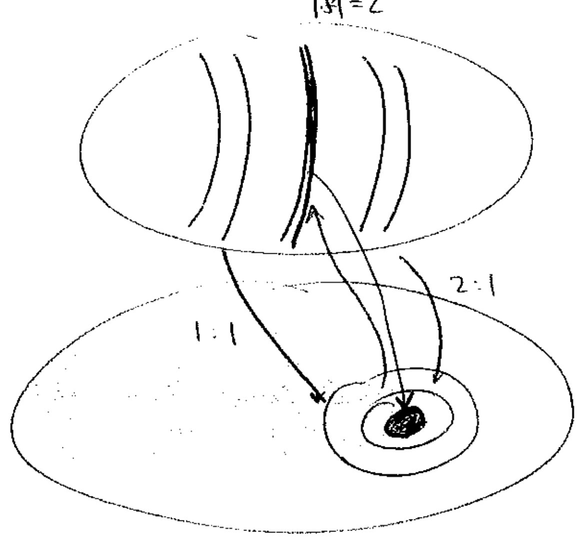
Conclusion:

2-adic modular fns $= \mathbb{Q}_2\langle f\rangle$

overconvergent " " $= \mathbb{Q}_2\langle \alpha f\rangle$

$$X_0(2) \longrightarrow X_0(1)$$

$|\alpha| = 2^8$



$\pi^{-1}(\infty) = \infty+0+0$ ①

$(j=1728)$ $\pi^{-1}(\text{order } 4) = 2$ pts ①

$\pi^{-1}(\text{order } 6) = 1$ pts ②

$(j=0)$

$$PSL_2(\mathbb{F}_3) \curvearrowright X_0(3) \xrightarrow{12} X_0(1)$$

$PSL_2(\mathbb{F}_3)$ $= 8\cdot 6$

$$-2 = 3(-2) + \sum_{\text{to } p} e_i - 1$$