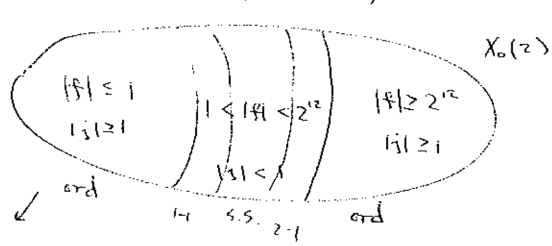
Feb 28. 2006. Kevin Buzzard. (E.V.), Tuesday. 6th lecture

Recall notation

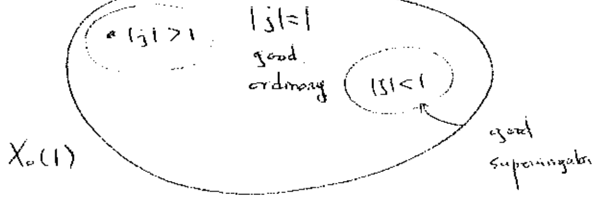
$$f = \frac{\Delta(q^2)}{\Delta(q)} \quad \text{an isomorphism} \quad f: X_0(2) \longrightarrow \mathbb{P}^1$$

$$\begin{tikzcd} X_0(2) \arrow[r, "f"] \arrow[d, "\text{forget (deg 3)}"] & \mathbb{P}^1 \arrow[d] \\ X_0(1) \arrow[r, "j"] & \mathbb{P}^1 \end{tikzcd}$$

$$\frac{1}{j} = \frac{f}{(1+256f)^3}$$



$p=2$



In particular,

$$\frac{1}{j} = f + \cdots \in \mathbb{Z}_2[[f]]$$

$$\therefore f = \frac{1}{j} + \cdots \in \mathbb{Z}_2[[\tfrac{1}{j}]]$$

Better: $\frac{2^8}{j} = 2^8 f + \quad \in \mathbb{Z}_2 [[2^8 f]]$

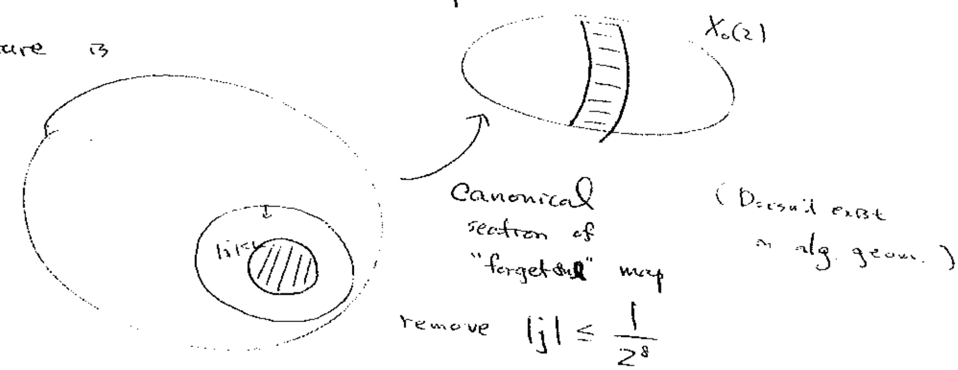
$\therefore\ 2^8 f = \frac{2^8}{j} + \quad \in \mathbb{Z}_2 [[\frac{2^8}{j}]]$

$\therefore$ even if $|j|$ is a little less than 1

there's a canonical f associated to j

i.e. a section of the "forget" map.

So picture is



Recall a 2-adic modular fctn is a (2-adic) holomorphic fctn on $X_0(1)^{ord}$ = closed disc $|j| \geq 1$ = closed disc $|f| \leq 1$

i.e. an element of $\mathbb{Q}_2\langle f \rangle$

An overconvergent 2-adic modular fctn is a fctn on $X_0(1)^{ord}$ that extends a little into the missing disc

e.g. choose t, $1 \leq t \leq 7$

& Consider the ~~ring~~ ring

$\mathbb{Q}_2 \langle 2^t f \rangle$

— any element here is overconvergent.

the bigger t is, the more overconvergent you are

Hecke ops

Classical modular forms come equipped with Hecke operators

$T_2, T_3, T_5, \ldots$

$\boxed{T_\ell \mathrel{\circlearrowright} M_k(\Gamma_0(1))}$

Do they extend to the p-adic setting?

Yes, & it's not too hard to see why

What are Hecke operators?

If a modular function is a "function on elliptic curves".

(Katz)

$E \longmapsto f(E)$

then $T_\ell f$ (ℓ prime) is $\frac{1}{\ell} \sum_{C \subseteq E} f(E/C)$

In our p-adic setting,

a p-adic modular function is a fn on ordinary elliptic curves   i.e. has either bad or good ordinary reduction.

& it's well-known that anything isogeneous to ordinary is ordinary

So formula makes sense! ($\frac{1}{\ell} \sum_{\substack{C \subseteq E \\ \text{ordinal}}} f(E/C)$)

$\therefore$ Get a bunch of continuous endomorphism of the space of p-adic modular functions.

& they all commute.

$|f| = \sup_{x \in X_0(1)^{\mathrm{ord}}} |f(x)| \in \mathbb{R}_{\geq 0}$

$\mathbb{Q}_p\langle T\rangle \ni \sum a_n T^n = g$

$|g| = \max |a_n|$

A nice thing to have, though would be one compact Hecke operator!

If $\varphi : V \longrightarrow V$ is cpt.

then we can start pulling off fin-dim subsp. of V

(generalized eigenspaces for φ. non-zero eigenvalue)

& they will be stable under all the other Hecke operators

I don't know any natural such things though

However, if we consider overconvergent forms, one does appear

Recall from last time

if $E/\mathbb{Q}_2$ had supersingular but not too supersingular reduction

then amongst the 3 subgps of order 2 one stuck out with respect to valuations of coordinates

Formal groups: "$z = \frac{x}{y}$"

parameter for the formal gp of elliptic curve

$y^2 =$ cubic of x.

& what I did last time was to show that the 3 points of order 2 in the formal gp of the curve were in 2 classes

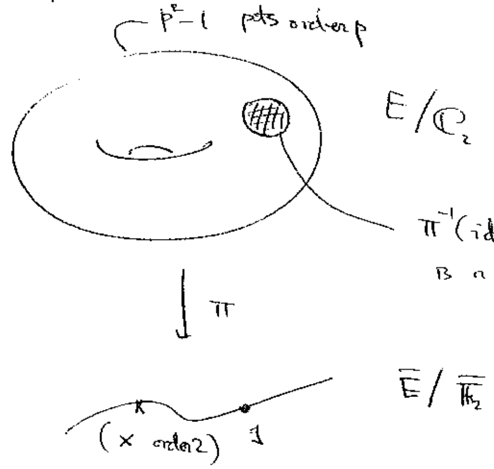
two had one norm, the third one (canonical one) had a different norm.

Formal gp $= \mathcal{O}_2[[Z]]$

$[2]Z =$ power series in Z

Zeros of power series

= pts of order dividing 2 in formal gp.



In ordinary case amongst the 3 pts of order 2
only one is in the disc

In supersingular case. all 3 pts of order 2 are in disc k

Ordinary case

$[2]Z = 2Z + \text{const}\, Z^2 + \cdots$

& (Newton polygon) const is a 2-adic unit.

($Z = \frac{-2}{a}$ is close to another root)

In non-ordinary case,

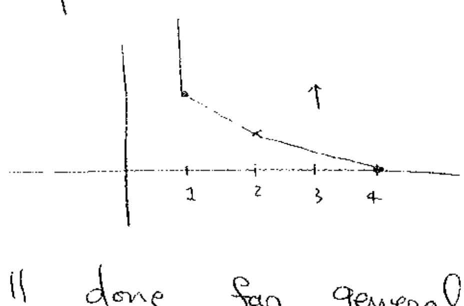
$$[2]Z = 2Z + aZ^2 + bZ^3 + cZ^4 + \cdots \qquad \mathcal{O}_2/2\mathcal{O}_2$$

c is now a unit

a is not a unit

but $2 \mid b$, because mod $2\mathcal{O}_{\mathbb{C}_2}$, $[2]Z$ is a fn of Z^2.

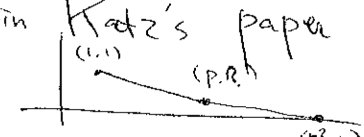
Newton polygon when $|a|<1$ but only just.



If $v(a) < \frac{2}{3}$.
can spot a canonical root

All done for general p in Katz's paper

$p > 2$ no harder than $p=2$



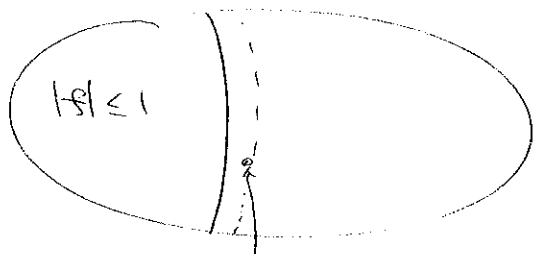
? = 0 ord. case

? > 0 but ? $< \frac{p}{p+1}$

get $p-1$ canonical sol's to $[p]z = 0$ & these are the canonical subgp.

Upshot: the canonical subgp of an ell. curve $E/\mathbb{C}_p$ depends only on the formal gp associated to E

$X_0(2)$



$|f| \le 1$

(E, C) C canonical

C will exist when $v(a)$ is small

Miracle: if l is prime, $l \neq 2$, then

E & E/D have isomorphic formal groups!

$D \subset E$ is a subgp of order l

$$\underbrace{E \xrightarrow{\varphi} E/D \xrightarrow{\varphi^\vee} E}_{\times l}$$

$\mathrm{End}(\text{formal gp}) \supseteq \mathbb{Z}_p = \mathbb{Z}_2$

& $l \in \mathbb{Z}_2^\times$

So the Hecke operators

T_l, $l \neq p$ don't change $|f|$ as long as $1 < |f| < \underset{\text{small thing}}{2^8}$

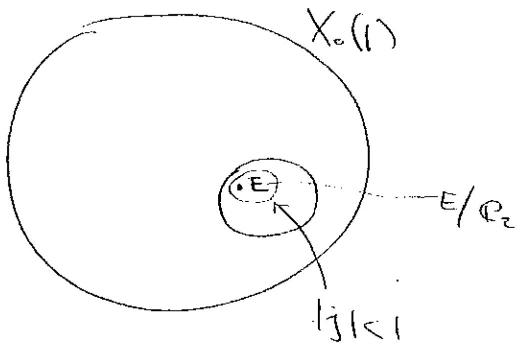
Conclusion:

T_l acts on $\mathbb{Q}_2\langle 2^t f\rangle$ for $1 \le t \le 7$ as well if $l \neq 2$.

Unfortunately, T_2 doesn't preserve $\mathbb{Q}_2\langle 2^t f\rangle$ if $1 \le t \le 7$

T_2 "makes things worse".

Can we see what's happening?



Say we know $j(E)$ & it has norm $1-\varepsilon$.

Let C be one of the subgps of E of order 2

What is $j(E/C)$?

Answer is given by the classical <u>modular polynomial</u>

$$\Phi_2(X,Y)$$

a polynomial deg 3 in each variable s.t $\Phi_2(j(\tau), j(2\tau)) \equiv 0$

$E = \mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}(2\tau)$

$C = \langle\tau\rangle$ order 2

$E/C = \mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}\tau$

<u>Recall</u>

$$\begin{aligned}\Phi_2(X,Y) = {} & X^3 + (-Y^2+1488Y-162000)X^2 \\ & + (1488Y^2+40773375\cdot Y+8748000000)X \\ & + (Y^3-162000Y^2+8748000000Y-157464000000000)\end{aligned}$$

Check sub. in Y = 2-adic unit

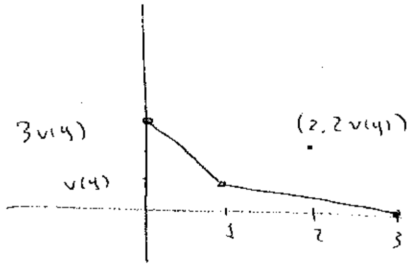
All 3 roots for X should be units. ✓

Now sub. in $Y = y \in \mathbb{Q}_2$

$1-\varepsilon = |y| < 1$

Hope: all 3 roots for X have $|X|=|Y|$ and T_2 should exist

Newton polygon



Conclusion:

if $v(j(E)) = c$, $c = \delta$, then $v(j(E/\overset{\text{order } 2}{C}))$ is either $2c$ (once) or $\frac{1}{2}c$ (twice)

$E/\mathbb{Q}_2$ s.s. reduction $v(j(E)) = \delta > 0$

3 subgps order 2, of which 1 is different

3 answers for $j(E/C)$, one of which is different.

Fact: Obvious guess is correct! The canonical subgp gives rise to a totally diffrent j

So in fact T_2 breaks up into 2 Hecke operators

$$(T_2 f)(E) = \frac{1}{2}\left(f(E/C_1) + f(E/C_2) + f(E/C_3)\right)$$

& if E is ordinary or "not too supersingular" then one of the Cs, say C_1 is canonical

Define $(Vf)(E) = \frac{1}{2} f(E/C_1)$

& $(Uf)(E) = \frac{1}{2}\left(f(E/C_2) + f(E/C_3)\right)$

U involves elliptic curves s.t. val'n of j-invariant has gone down.

$\therefore$ U (but not V) will induce an endomorphism of $\mathbb{Q}_2\langle 2^t \cdot f\rangle$

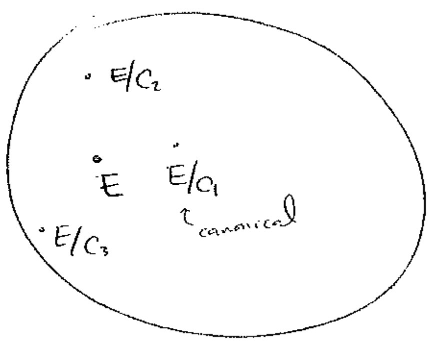
(& all this works for general N, p)

(play around with power series $[p]Z$)

In fact more is true. Val^n of j-invariant was being divided by 2 when we applied U

Pictorially

$|j| < 1$



& we in fact see that U is a conti. map

$$\mathbb{Q}_2\langle 2^t\cdot f\rangle \longrightarrow \mathbb{Q}_2\langle 2^{2t}\cdot f\rangle \hookrightarrow \mathbb{Q}_2\langle 2^t\cdot f\rangle$$

for $t \in \{1,2,3\}$

More generally, U is a continuous map from "r-overconvergent p-adic modular fns" to "r'-overconvergent p-adic modular fns".

U "increases overconvergence"

Consequence:

The induced map $U: \mathbb{Q}_2\langle 2^t\cdot f\rangle \longrightarrow \mathbb{Q}_2\langle 2^t\cdot f\rangle$ $(1\le t\le 3)$ is cpt.

Pf. U is composition of a conti. map & the inclusion

$\mathbb{Q}_2\langle 2^{2t}\cdot f\rangle \hookrightarrow \mathbb{Q}_2\langle 2^t\cdot f\rangle$ & this is cpt!

cpt $\circ$ cts $=$ cpt.

$\mathbb{Q}_p\langle T\rangle$ has an ON basis

$1, T, T^2, T^3, \ldots$

& w.r.t the obvious basis $\qquad 2^t \times 2^t\cdot f$

the inclusion $\mathbb{Q}_2\langle 2^{2t}\cdot f\rangle \hookrightarrow \mathbb{Q}_2\langle 2^t\cdot f\rangle$

has matrix $\begin{pmatrix} 1 & & & \\ & 2^t & & \\ & & 2^{2t} & \\ & & & 2^{3t} \end{pmatrix}$

- bdd entries
- All $< \varepsilon$ under hori. line.

Remark

$U = U_2$ on $X_0(2)$, $\quad U_2 f(E,C) = \frac{1}{2} \sum_{D \neq C} f(E/D, C)$

(not canonical)

Effect on q-expansions

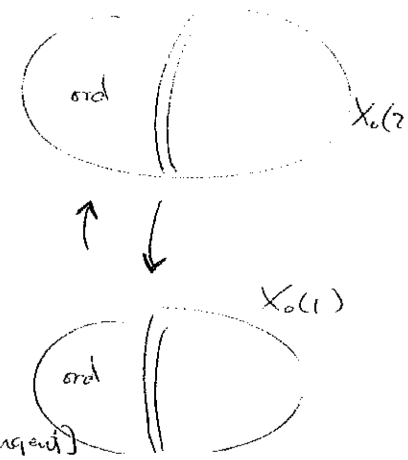
An elementary computation with Tate curves gives that if $f = \sum a_n q^n$ is a 2-adic modular form, then Uf has q-expansion $\sum a_{2n} q^n$.

[More generally, general p, $U(\sum a_n q^n) = \sum a_{np} q^n$, → endo. of p-adic, & r-overconv. p-adic modular ftns

& $V(\sum a_n q^n) = \text{cst} \times \sum a_n q^{pn}$.

To make our life easier, let's redefine V

so $V(\sum a_n q^n) = \sum a_n q^{2n}$.



In general, $V(\sum a_n q^n) = \sum a_n q^{np}$ is an endomorphism of space of p-adic modular ftns (not overconvergent)