Apr 6. 2006. Thursday. Kevin Buzzard. 14th lecture.

Recall: (1) $F/\mathbb{Q}_p$ fin.

if $\chi_1, \chi_2 : F^\times \to \mathbb{C}^\times$ are cts

then define $I(\chi_1, \chi_2) = \left\{ f : GL_2(F) \to \mathbb{C} \text{ locally const s.t. } f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g\right) = \chi_1(a)\cdot\chi_2(d)\left|\frac{a}{d}\right|^{\frac{1}{2}} f(g) \right\}$

Rmks about $I(\chi_1, \chi_2)$.

if $\chi_1 = |\cdot|^{-\frac{1}{2}}$ & $\chi_2 = |\cdot|^{+\frac{1}{2}}$, then this cancells out $\left|\frac{a}{d}\right|^{\frac{1}{2}}$ factor.

& $I(|\cdot|^{-\frac{1}{2}}, |\cdot|^{\frac{1}{2}}) \supset$ const fns.

a $GL_2(F)$-inv. subspace

Define $St = I(|\cdot|^{-\frac{1}{2}}, |\cdot|^{\frac{1}{2}}) / \text{cstn}$

(2) One checks if $\chi : F^\times \to \mathbb{C}^\times$

then $I(\chi_1, \chi_2) \otimes (\psi \circ \det) = I(\chi_1\psi, \chi_2\psi)$

As a conseq. $I(\chi_1, \chi_2)$ is <u>not</u> <u>irreducible</u> whenever $\chi_1/\chi_2 = |\cdot|^{-1}$.

(3) There's a notion of a dual of a smooth adm. repn

— take abstract dual & then smooth vectors in it.

$\bigoplus_{i \in \mathbb{N}} \mathbb{C}$, $\prod_{i \in \mathbb{N}} \mathbb{C}$

(4)

Turns out that the

Russian Math Survey

Bernstein - Zelevinsky

<u>dual</u> of $I(\chi_1, \chi_2)$ is $I(\chi_1^{-1}, \chi_2^{-1})$

<u>Rk</u>: would not be true if we used "un-normalized" induction

$G \supset B$

G/B

Haar measure

$|\cdot|$, involved in answer

Conseq: $I(\chi_1, \chi_2)$ not irreducible, if $\frac{\chi_1}{\chi_2} = |\cdot|$

& in fact has 1-dim quotient

⑤ If $\chi_1 \neq \chi_2 \times |\cdot|^{\pm 1}$ then $I(\chi_1, \chi_2)$ is irreducible. (ex. Jacquet-Langlands)

Let's define $PS(\chi_1, \chi_2) := I(\chi_1, \chi_2)$ in this case

St is also irreducible

(as are all twists of St) & the 1-d rep'n $\psi \circ \det$ are irreducible

The only isom between them are $PS(\chi_1, \chi_2) = PS(\chi_2, \chi_1)$

If $\chi : F^\times \to \mathbb{C}^\times$ is cts.
define $PS(\chi |\cdot|^{\frac{1}{2}}, \chi|\cdot|^{-\frac{1}{2}}) = PS(\chi|\cdot|^{-\frac{1}{2}}, \chi|\cdot|^{\frac{1}{2}}) = \chi \circ \det$

① PS
② $St \otimes \chi$ (special)
③ The rest. These are called supercuspidal
Examples are give by $BC(\psi)$ ("base change")
but if res char of F is 2, there are more

On the other hand, there's a classification of φ-semi-simple 2-dim. Weil-Deligne rep'n of W_F.

the idea

$$\begin{array}{ccccccccc} 0 \to & I & \to & Gal(\bar{F}/F) & \to & \hat{\mathbb{Z}} & \to 0 \\ & \| & & \cup & & \uparrow & \\ 0 \to & I & \to & W_F & \to & \mathbb{Z} & \to 0 \end{array}$$

Topology W_F s.t I is open

Define $\|\cdot\| : W_F \to \mathbb{C}^\times$

by defining $\|\cdot\| \,|_I = 1$
& $\|\text{Geom. Frob}\| = \dfrac{1}{\#(\text{residue field})}$

A Weil-Deligne rep'n of W_F is a f.d $\mathbb{C}$-vector sp V (Geom. Frob $=$ Arith Frob^{-1})
& cts map $\rho : W_F \to Aut(V)$ discrete top
& $N \in End(V)$ nilpotent
$\varphi N = pN\varphi$ s.t $\rho(w) \cdot N = \|w\| \cdot N \cdot \rho(w)$

The pair (ρ, N) is φ-semi simple if ρ is semi-simple

Classification of φ-ss WD reps

① Reducible ones $\chi_1 \oplus \chi_2$ $N=0$

$$\chi_i : W_F^{ab} \longrightarrow \mathbb{C}^\times$$
$\| \leftarrow$ LCFT
$F^\times$ geom Frob $\to$ unifor.

② $\rho = \begin{pmatrix} \chi\|\cdot\|^{\frac{1}{2}} & 0 \\ 0 & \chi\|\cdot\|^{-\frac{1}{2}} \end{pmatrix}$. $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

③ ρ irreducible. & $N=0$.

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}$

e.g. if $\chi : W_K \to \mathbb{C}^\times$ K/F quad $\sigma : K \to K$
$\psi \neq \psi \circ \sigma$,

then $\mathrm{Ind}_{W_K}^{W_F} \psi$ is irreducible

If res char $F \neq 2$, then all irreducible reps are of the form

Local Langlands matches up
φ - semi-simple n-dim WD reps with irred smooth reps of $GL_n(F)$

For $n=2$, this is a thm of Jacquet, Langlands, Kutzko ($\ell = 2$)

There is a way to match things up that's natural from a local pt of view (L-factor, ε-factor)

In this case

$\mathbf{PS}(\chi_1, \chi_2) \longleftrightarrow \chi_1 \oplus \chi_2$

$\mathrm{Sp} \otimes \chi \longmapsto \begin{pmatrix} \chi\cdot|\cdot|^{+\frac{1}{2}} & 0 \\ 0 & \chi\cdot|\cdot|^{-\frac{1}{2}} \end{pmatrix}$. $N \neq 0$.

SC $\longleftrightarrow$ irred

For local-global compatibility, another normalization is better.

$\mathbb{C} \cong \overline{\mathbb{Q}}\{x_i\}_{i \in I}$ over $\overline{\mathbb{Q}}$

Local - global compatibility.

Let f be a modular form

Fix an isom $\overline{\mathbb{Q}_p} \cong \mathbb{C}$

Assume f is a cuspidal eigenform.

Then $\exists$ Gal rep.

$$\rho_f : G_{\mathbb{Q}} \longrightarrow GL_2(\overline{\mathbb{Q}_p})$$

If f has level M & ℓ is a prime

$\ell \nmid Mp$. then $\rho_f(\mathrm{Frob}_\ell)$ has char poly $X^2 - a_\ell X + \ell^{k-1}\chi(\ell)$

a_ℓ = e. val of T_ℓ on f

χ = char of f

Non-triv.

Generalization

Let ℓ be any prime, $\ell \neq p$

($\ell | M$ is o.k.)

$\rho_f|_{D_\ell} \cong$?

A. construction of Grothendieck associated to a cts rep

$\sigma_\ell : Gal(\overline{\mathbb{Q}}_\ell / \mathbb{Q}_\ell) \longrightarrow GL_n(\overline{\mathbb{Q}}_p)$

a Weil-Deligne rep'n.

$$\left(\begin{array}{l} \rho : W_{\mathbb{Q}_\ell} \longrightarrow GL_n(\overline{\mathbb{Q}}_p \cong \mathbb{C}) \\ + N \end{array} \right)$$

Rmk: if σ_ℓ is semi-simple

then $N = 0$ & ρ "is" $\sigma_\ell|_{W_{\mathbb{Q}_\ell}}$

if $\sigma_\ell|_I$ is non-semi-simple

then $N \neq 0$ to reflect this.

So $\underbrace{f \to \rho_f}_{\text{Global}} \to \rho_f|_{D_\ell} \longrightarrow (\rho, N)$ is a Weil-Deligne rep'n of $W_{\mathbb{Q}_\ell}$ associated to f.

On the other hand. if M = level of f

then $f \in H^0(Y_1(M)_{/\mathbb{C}}, \omega^{\otimes k})$ & $Y_1(M)(\mathbb{C})$ has an adelic interpretation as follows

Fact. $GL_2(\mathbb{Q})\ GL_2(\hat{\mathbb{Z}}) = GL_2(\mathbb{A}_f)$

$\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p \quad \mathbb{A}_f = \mathbb{A}^\infty = \hat{\mathbb{Z}} \otimes \mathbb{Q}$

(& in fact $GL_2(\mathbb{Q})\, K_1(M) = GL_2(\mathbb{A}_f)$, $\forall M$)

If $K_1(M)$ = subgp of $GL_2(\hat{\mathbb{Z}})$ consisting of matrices $\equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$ mod M.

& if $GL_2(\mathbb{Q}) \subset \underset{GL_2(\mathbb{R})/SO_2(\mathbb{R})\cdot\mathbb{R}^*}{\mathfrak{h}^\pm} = \{x + iy \in \mathbb{C} : y \neq 0\}$

then $$\underset{\text{diagonal}}{GL_2(\mathbb{Q})} \Big\backslash \Big(\underset{GL_2(\mathbb{Q})\, K_1(M)}{\overset{}{GL_2(\mathbb{A}_f)}} \times \mathfrak{h}^\pm \Big) \Big/ \underset{\substack{\text{acts on } \mathfrak{h}^\pm \\ \text{on } GL_2(\mathbb{A}_f)}}{K_1(M)}$$

$$GL_2(\mathbb{Q}) \Big\backslash GL_2(\mathbb{Q}) \cdot K_1(M) \times \mathfrak{h}^\pm \Big/ K_1(M)$$

$$GL_2(\mathbb{Q}) \cap K_1(M) \Big\backslash K_1(M) \times \mathfrak{h}^\pm \Big/ K_1(M)$$

$$= \quad GL_2(\mathbb{Q}) \cap K_1(M) \Big\backslash \mathfrak{h}^\pm \quad = \quad \Gamma_1(M) \backslash \mathfrak{h} = Y_1(M)(\mathbb{C})$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z})$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \text{ mod } M$$

So $\quad Y_1(M)(\mathbb{C}) = GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}_f) \times \mathfrak{h}^\pm / K_1(M)$

& if $g \in GL_2(\mathbb{A}_f)$

then right multiplication by g induces a map

$$\begin{array}{c} GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}_f) \times \mathfrak{h}^\pm / K \\ \downarrow r_g \\ GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}_f) \times \mathfrak{h}^\pm / g^{-1}Kg \end{array} \qquad (K: \text{any cpt open in } GL_2(\mathbb{A}_f))$$

The spaces $GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}_f) \times \mathfrak{h}^\pm / K$, K any open cpt,

are generalizations of the modular curves $Y_1(M)$

& if K is suff. small they have sheaves $\omega^{\otimes k}$ on them

& $\exists$ natural isom

$$r_g^* \omega^{\otimes k} \cong \omega^{\otimes k}$$

Idea: think of $GL_2(\mathbb{Q}_\ell) \subseteq GL_2(\mathbb{A}_f)$ & given $f \in H^0(Y_1(M), \omega^{\otimes k})$

One can look at the ∞-dim $\mathbb{C}$-v. space spanned by $r_g^* f$, $g \in GL_2(\mathbb{Q}_\ell)$ cuspidal

$GL_2(\mathbb{Q}_\ell)$ acts on modular curve $\Rightarrow$ acts on the space

So this space called $\pi_{f,\ell}$ has an action of $GL_2(\mathbb{Q}_\ell)$

If f is an eigen form, then $\pi_{f,\ell}$ is smooth & irreducible

$$\begin{array}{ccc} f & \longrightarrow & (\rho, N) \quad \left(\begin{array}{l}\text{Global} \\ \text{via Gal rep'n}\end{array}\right) \\ \downarrow & & \\ \pi_{f,\ell} & & \end{array}$$

Q) Does $\pi_{f,\ell}$ become associated with (ρ, N) via local Langlands

reek — Not if one use the usual normalization

r_ℓ — But one is only by a dual & a twist

ρ: red & $N=0 \iff \pi_{f,\ell}$ is PS

$N \neq 0 \iff \pi_{f,\ell}$ special

ρ: irred $\iff \pi_{f,\ell}$ s.c.

Simplest case $\ell \nmid M$

turns out that $\pi_{f,\ell}$ is isom. to $PS(\chi_1, \chi_2)$

$$\chi_1, \chi_2 : \mathbb{Q}_\ell^\times \longrightarrow \mathbb{C}^\times \text{ trivial on } \mathbb{Z}_\ell^\times$$

& $\frac{\chi_1(\ell)}{\sqrt{\ell}}, \frac{\chi_2(\ell)}{\sqrt{\ell}}$ are the roots of

$$X^2 - a_\ell X + \ell^{k-1} \chi(\ell)$$

but $\rho_f|_{D_\ell} \cong \begin{pmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{pmatrix}$ $\quad \psi_1, \psi_2$: unram

$\psi_1(\frac{1}{\ell}), \psi_2(\frac{1}{\ell})$ are roots of $X^2 - a_\ell X + \ell^{k-1} \chi(\ell)$

$$\{\chi_1, \chi_2\} \neq \{\psi_1, \psi_2\}$$

but we're only done by a dual & a twist

<u>Thm</u>. This is true in general

(Deligne, Langlands, Carayol)

If you use the "r_e" normalization of local Langlands

then $\pi_{f,\ell}$ becomes identified with (ρ, N)

Practical consequences.

If f is a new **form** level $M = N \cdot \ell^n$ $n \geq 0$, $\ell \nmid N$

& ρ_f is the associated p-adic Gal rep., $\ell \neq p$

then $\rho_f | I_\ell$ is non-semi simple $(\Leftarrow \rho_f(I_\ell)$ has infinite image$)$

$\iff \pi_{f,\ell}$ is special.

If $\rho_f(I_\ell)$ is finite

then $\rho_f|_{D_\ell}$ is irreducible $\iff \pi_{f,\ell}$ is supercuspidal.

& $\underset{\rho_f|D_\ell}{\wedge}$ reducible $\iff \pi_{f,\ell}$ is PS.

In practice, f has level $N \cdot \ell^n$ & we'll know a_ℓ = coeff of q^ℓ

& char χ of f = $\underset{N\text{-part}}{\chi_N} \cdot \underset{\ell\text{-part}}{\chi_\ell}$

Thm

If $n=0$. ($\ell \nmid$ level)

then $\rho_f | D_\ell$ is unram. & cp of $\mathrm{Frob}_\ell = x^2 - a_\ell x + \ell^{k-1} \chi_N(\ell)$

If $n=1$, but χ_ℓ is trivial.

then $\pi_{f,\ell}$ is unram. twist of St.

a_ℓ tells you the twist $\qquad a_\ell^2 = \ell^{k-2} \cdot \chi_N(\ell)$

& $\rho_f | D_\ell = \begin{pmatrix} \lambda(\ell \cdot a_\ell) & * \\ 0 & \lambda(a_\ell) \end{pmatrix}$

where here $\lambda : D_\ell \to \overline{\mathbb{Q}}_p^\times$ is the unram. char sending

$\lambda(x) = \text{arithmetic Frob} \longmapsto x$ & $*$ is ramified ($\rho_f|_I$ non-semi-simple)

(rk: ratio of characters is the cyclotomic char & $* \neq 0$ can happen)

e.g. $T_p\left(\overline{\mathbb{Q}}_\ell^\times / \langle q \rangle\right)$ $q \in \mathbb{Q}_\ell^\times$, $|q| < 1$ $\qquad$ 1-d sub cyclo, 1-d quot. unram, $* \neq 0$

If $n \geq 1$ & $\mathrm{cond}(\chi_\ell) = \ell^n$ (= as big as it can be)
then $\pi_{f,\ell}$ must be $PS(\chi_1, \chi_2)$ with precisely of the χ_i : unramified

In this case, $a_\ell \neq 0$.

& $\rho_f | D_\ell = \lambda(a_\ell) \oplus \lambda(\ell^{k-1} \chi_N(\ell)/a_\ell)\chi_\ell$

In all other cases. $\underline{a_\ell = 0}$ & it's a little trickier to compute $\pi_{f,\ell}$.

$\left.\begin{matrix}\text{St} \otimes \text{ram} \\ \text{PS.(ram, ram)} \\ \text{SC}\end{matrix}\right) \; a_\ell = 0.$

$\det \rho_f = \mathrm{cyclo}^{k-1} \cdot \chi$

$\det \rho_f|_{D_\ell} = \mathrm{cyclo}^{k-1} \cdot \chi_N \cdot \chi_\ell$