

f : cuspidal eigenform

(Thursday)

17th lecture

$$\rho_f : G_{\mathbb{Q}} \xrightarrow{\text{irreducible}} GL_2(\overline{\mathbb{Q}}_p)$$

$$\downarrow$$

$$\Pi_{\mathbb{F}, p}$$

$$\cup$$

$$GL_2(\mathbb{Q}_p)$$

local Langlands
 \updownarrow
 Weil-Deligne rep'n
 \updownarrow
 φ, N (Gal)-modules

$$\downarrow$$

$$\rho_f | D_p$$

Target filtration \leftarrow $D_{\text{pst}}(\rho_f | D_p) =$ weakly adm. filtered φ, N (Gal)-module

$$[D_{\text{st}}(\rho_f | D_p) = \text{w.a. fil. } \varphi, N\text{-module}]$$

Let me stick to the case where either

- (1) $p \nmid$ level of f (conductor)
- (2) $p \parallel$ level of f (conductor) & charact of f has conductor prime to p

In the last lecture

I unravelled things explicitly in cases 1 & 2.

In case 2 if you forget the filtration, you're stuck w/ many rep'n.

\leftarrow \mathbb{Z} -invariant of the situation $\in \overline{\mathbb{Q}}_p$

In case 1, a miracle occurs:

IF $D = (\varphi, N)$ -module, then $N = 0$.

& φ has e.val. the roots of $X^2 - a_p X + p^{k+1} \chi(p)$.

& IF these roots are distinct, you can diagonalize φ

$$\varphi = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \subset \overline{\mathbb{Q}}_p e_1 \oplus \overline{\mathbb{Q}}_p e_2$$

& up to isom. only

[Neeve]

3 lines $\langle e_1 \rangle, \langle e_2 \rangle$ & any other $q + \delta e_2$

Furthermore, one isn't adm. & if $|a_p| > 0$ then 2 aren't adm.

So only one choice for filtration.

& $\Pi_{\mathbb{F}, p}$ & \mathbb{K}

determined by $\rho_f | D_p$

Rmk: This observation is essentially a coincidence = 136

It fails in alm. all other situations.

e.g. for Hilbert mod. forms

one replace \mathbb{Q}_p by a fin. ext'n K of \mathbb{Q}_p
 $\&$ if $K \neq \mathbb{Q}_p$, then always $n > 1$ w.r. filtration

$$K^s / \mathbb{Q}_p^s$$

Exercise even fails for $GL_3(\mathbb{Q}_p)$:

$$N=0$$

$$D = 3\text{-dim'l} / \mathbb{Q}_p$$

$\varphi = \begin{pmatrix} \alpha & & \\ & \beta & \\ & & \gamma \end{pmatrix}$ The Hodge theory is a choice of a plane
 i.e. e_1, e_2 $p \subseteq D$ & a line $L \subseteq p$.

If p & L are "in general position" w.r.t φ

then the filt'n is weakly admissible.

\therefore w.a. filt'n form an open in the flag variety.

(2-dim'l choice of possibilities for p
 1-dim'l choice for L

$\therefore (p, L) \in 3\text{-d. space}$

$$\begin{aligned} e_1 &\rightarrow \lambda e_1 \\ e_2 &\rightarrow \mu e_2 \\ e_3 &\rightarrow \gamma e_3 \end{aligned}$$

The torus in $PGL_3(\mathbb{Q}_p)$ acts on this space but this is only 2-dim'l w/ many orbits.

Back to the dictionary

Say f is a modular form of level N prime to p .

$T_p f = a_p f$, f has a char. χ .

$\&$ assume $\chi(p) = 1$.

Assume that $X^2 - a_p X + p^{k+1}$ has distinct roots.

Hodge Thy cannot see

$$a_p \text{ unit}$$

$$P_R = \begin{pmatrix} \text{cyc}^{k+1} & & \\ & \circ & \\ & & \cdot \end{pmatrix}$$

$$f \rightarrow \pi_{f,p} \rightarrow (P_R / D_p)^{ss}$$

Let's now reduce P_R / D_p mod p . & we get $\overline{P}_{k,q_p} = \mathbb{Q}_{q_p} \rightarrow GL_2(\mathbb{F}_p)$
 a semi-simple mod p Galois rep'n

$$\pi_{F,p} = \text{Ind}(x_1, x_2) \quad \begin{cases} x_1|_{Z_p^*} = x_2|_{Z_p^*} = 1 \\ x_1(p) = \alpha \\ x_2(p) = \beta \end{cases}$$

Easy Cases

$$\overline{\mathbb{Q}}_p \cong \mathbb{C}$$

- If $k=1$, then $\rho_F : G_{\mathbb{Q}} \rightarrow GL_2(\mathbb{C})$
 ρ_F is unram. at p , $\rho_F(\text{Frob}_p)$ is semi-simple
 (Hida) with e.val. the roots of $X^2 - a_p X + 1$.

- $|a_p| = 4$: $\rho_F|_{D_p^{ss}} = \text{cyclo}^{k+1} \cdot \chi(\alpha^{-1}) \oplus \chi(\alpha)$
 $k \geq 2$ where $\alpha = \text{unit root of } X^2 - a_p X + 1$

- For cases : $k \geq 2$ & $|a_p| < 1$.
 Set $v = v(a_p)$ ($v(p) = 1$)

If $2 \leq k \leq p-1$ & $v > 0$

then $\overline{\rho}_{k,q}$ is irreducible & I can tell you what it is.

[Reminder of semi-simple Galois reps]

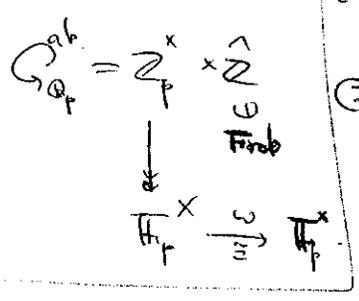
$$\overline{\rho} : G_{\mathbb{Q}_p} \rightarrow GL_2(\overline{\mathbb{F}}_p) \text{ with determinant } \omega^{k-1}$$

$\omega = \text{mod } p \text{ cyclo char.}$ (from modular forms)

① Reducible case, $\overline{\rho} = \chi_1 \oplus \chi_2$

$$\chi_1 = \omega^a \cdot \chi(\alpha) \quad \begin{matrix} \text{arithmetic} \\ \text{Frob.} \end{matrix} \rightarrow \alpha \text{ unramified char.} = \chi(\alpha)$$

$$\chi_2 = \omega^{p-1-a} / \chi_1 \quad \text{where } 0 \leq a < p-1 \text{ \& } \alpha \in \overline{\mathbb{F}}_p^*$$



② Irred. case

wild inertia in $G_{\mathbb{Q}_p}$ is pro- p

& normal, so invariants under wild

inertia are non-zero & an int-submod.

= wild inertia acts trivially

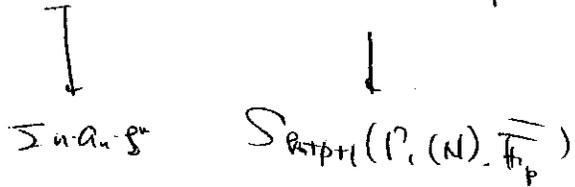
If $v \geq 1$, then $\overline{P}_{k, a_p} = \omega \otimes$ unram semi-simple repr
with Frobp e.vals
the roots of $x^2 - \left(\frac{a_p}{p}\right)x + 1$

Rmk: If $0 < v < 1$, then f lies on a cpnt of eigen curve
(Question)? that also contains some form g .
 $w(g) = w(f) - (p-1)$
 $v(a_p(g)) = v(a_p(f))$

If $v \geq 1$,
then $\overline{f} = \theta g$

$a_p(g) = \frac{a_p(f)}{p} \cdot g \text{ wt } 1$

$\Sigma a_n g^n \in S_{k+1}(N, \overline{\chi}_p)$ Katz.

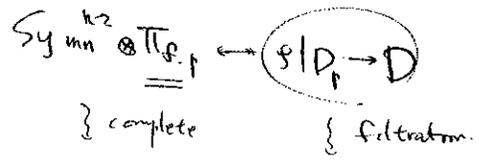


If f is an eigenform, then $\rho_{\text{GF}} = \omega \otimes \rho_f$

If $k+3 \leq k \leq q$

then $0 < v < 1$. $\overline{P}_{k, a_p} = I(k-1-(p-1)) = I(k-p)$

$v=1$: if $d = \left(\frac{a_p}{p}\right) \times (k-1)$, then \overline{P}_{k, a_p} is reducible,
& one character is $\omega \cdot \chi(d)$.



$v > 1$: $I(k-1)$

General Thm of (Berger-Li-Zhu)

If $k \geq 2$ & $v > \left\lfloor \frac{k-2}{p-1} \right\rfloor$, then $\overline{P}_{k, a_p} = I(k-1)$

unless $p+1 \mid k-1$, in which case it's ω_{Frobp} \otimes unram s.s. repr with Frobp e.val $\neq i$

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Rk, so far, answer has depended essentially only on $v(q)$

However, if $\pi_{f,p} = \text{twist of Steinberg}$

then $\overline{\rho}_{f,p}$ is not determined by $\pi_{f,p}$.

& There are pts in the eigencurve:

↔ newforms of level prime to p & have same slope

e.g. if $k=3$ & f is new @ p

$$\& v(a_p) = \frac{1}{2} = \frac{k-2}{2}$$

There's trouble in semi-stable case

Hence if $k = 3 + (p-1)p^n$ & $v(a_p) = \frac{1}{2}$,

$\overline{\rho}_{k,a_p}$ will depend on more than $v(a_p)$

If $k = 2p+1$ then

$v=0$: reducible. (char is $\lambda(\overline{\rho}_f)$)

$0 < v < \frac{1}{2}$: mod.

$$I(k-1-2(p-1)) = I(2)$$

$v = \frac{1}{2}$: need to know more!

Need to know: $W = \text{val}'_n \left(\frac{a_p^2 + p}{x(p)} \right)$

If $W < \frac{3}{2}$, then $\overline{\rho}_{k,a_p} = I(2)$

If $W \geq \frac{3}{2}$, then set $b = \frac{a_p^2 + p}{2pa_p}$

$$\overline{b} \neq 0 \iff W = \frac{3}{2}$$

$\overline{\rho}_{k,a_p}$ is $\omega \otimes$ unram rep with Frob eval. the roots of $x^2 - bx + 1$ ($\chi(p) = 1$)

If $v > \frac{1}{2}$, it's $I(k-1) = I(2p) \cong I(2)$

$v < \frac{1}{2}$, $I(k-1-2(p-1)) = I(2)$

$k = 2p+2$. Answer depends only on v ! (computationally)

$0 < v < 1 : I(k-1-2(p-1)) = I(3)$

$v = 1 : \text{reducible, one char } \omega \lambda \left(\frac{a_p}{p} \right)$

$1 < v : I(k-1-(p-1)) = I(p+2)$

$2p+2 \leq k \leq 3p-1$

Answer depends only on v (& on $\left(\frac{a_p}{p} \right)$ if $v = n$)

$k = 3p : \text{trouble @ } v = \frac{1}{2}$

$k = 3p+1 : \text{" @ } v = 1$

$k = 3p+2 : \text{" @ } v = 1\frac{1}{2}$

$p \geq 5, 3p+3 \leq k \leq 4p-2$ only depends on v

$4p-1, \dots, 1+p+3 : \text{trouble @ } v = \frac{1}{2}, 1, 1\frac{1}{2}, 2, 2\frac{1}{2}$

$k = p^2 : \text{trouble everywhere.}$

$a_p^2 + p$

$2p+4 \leq k \leq 3p-1 \quad \& \quad v = 2$

\overline{p}_{k, a_p} reducible & one char is $\omega^2 \cdot \lambda \left(\frac{(k-1)(k-2)}{2} \left(\frac{a_p}{p^2} \right) \right)$
