April 25. 2006. Tuesday. Kevin Buzzard (18th) lecture

(18-th lecture)

if $k \geq 2$ an integer & $a_p \in \overline{\mathbb{Q}}_p$. $|a_p| < 1$

$$a_p^2 \neq 4 \cdot p^{k-1}$$

then the φ-module

$D = \overline{\mathbb{Q}}_p^2$ + semi-simple φ-action

char poly $X^2 - a_p X + p^{k-1}$

has a unique wa filtration s.t $\mathrm{Fil}^0 = D$

$\mathrm{Fil}^1 = \text{line} = \mathrm{Fil}^{k-1}$

$\mathrm{Fil}^k = 0.$

$\Rightarrow$ Get a Gal rep'n V_{k, a_p}

Last time I told you that what I knew about $\overline{V_{k,a_p}}$

a semi-simple rep'n $G_{\mathbb{Q}_p} \to GL_2(\overline{\mathbb{F}}_p)$

It's a thm or a conj.

as to what $\overline{V_{k,a_p}}$ is for k "small"

When $k \geq p^2$ or so. it's a bit of mystery

Apart from Berger + Berger-Li-Zhu's result

Thm. If $v(a_p) > \left\lfloor \frac{k-2}{p-1} \right\rfloor$ & if $p+1 \nmid k-1$

then $\overline{V_{k,a_p}} = I(k-1)$ (i.e. $\bar{\rho}|I = \begin{pmatrix} \omega_2^{k-1} & 0 \\ 0 & \omega_2^{p(k-1)} \end{pmatrix}$)

If $p+1 \mid k-1$ & $v(a_p) > \left\lfloor \frac{k-2}{p-1} \right\rfloor$

then $\overline{V_{k,a_p}} = \omega^{\frac{k-1}{p+1}} \otimes$ unram semi-simple with Frob evals $\pm i$

Conj. If $p > 2$ & k is even. & $v(a_p) \notin \mathbb{Z}$.

then $\overline{V_{k,a_p}}$ is irreducible.

Rk: this implies that the p-adic val'ns of evals of

U_p G $S_k(\Gamma_0(p))$ are integers

for $2 < p < 59$. & all k.

One last remark about

$\Pi_{f,p}$ vs $\rho_f|D_p$.

I thought Satake's thm is the end of the story, but Breuil knew better.

Breuil realized there was a possibility of "inserting the Hodge theory into $\Pi_{f,p}$."

$\Pi_{f,p}$ is a countably infinite-dim. v. sp / $\mathbb{C}$ + smooth action of $GL_2(\mathbb{Q}_p)$.

Fix $\overline{\mathbb{Q}_p} \cong \mathbb{C}$.

The "Hodge" theory is 2 facts

① where filtration jumps $(0, k-1)$

② the line

The observation due to Breuil

is that if we consider the "locally alg" rep'n.

$$\Pi_{f,p} \underset{\overline{\mathbb{Q}}_p}{\otimes} \left(\mathrm{Sym}^{k-2} (\overline{\mathbb{Q}}_p^{\,2}) \right)^*$$

↑

f.d rep'n of $GL_2(\mathbb{Q}_p)$ — the data equivalent to the jumps

then, roughly speaking, the lines on the Galois by side should somehow correspond to

top. irred. "admissible unitary completions" of $\underline{\Pi_{f,p} \otimes (\mathrm{Sym}^{k-2} \overline{\mathbb{Q}}_p^{\,2})^*}$.

⇕

equiv. classes of certain $GL_2(\mathbb{Q}_p)$ - inv. norms on this guy.

φ, N - module

+ alg. rep'n ⟺ discrete data = jump

↓

$\Pi \times$ alg.

Known for $GL_2(\mathbb{Q}_p)$

is unram. PS case. there's a unique w.a. filtration $(|a_p| < 1)$

& a unique unitary norm with all the properties.

In Steinberg case, ∞ly many lines parametrized by some $\mathcal{L} \in \overline{\mathbb{Q}}_p$

& he's written down ∞ly many inequivalent norms. parametrized by $\mathcal{L} \in \overline{\mathbb{Q}}_p$.

completion $\neq 0$ uses $(\varphi$-$\Gamma)$-modules & $\mathcal{B}$ functor $(\varprojlim_{\psi} D)^b$.

Serre's Conjecture

i.e. mod p local / global local Langlands etc

I told you the following stuff already:

If p prime, $p \nmid N$.

& $f \in S_k(\Gamma_1(N), \overline{\mathbb{F}}_p)$ is an eigenform of character $\qquad H^0(X_1(N), \omega^{\otimes k}(\text{cusps}))_{\overline{\mathbb{F}}_p}$

$\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \overline{\mathbb{F}}_p^\times$.

then there's a mod p Galois rep'n $\rho_f : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{F}}_p)$
semi-simple

& ρ_f is unram at l if $l \nmid Np$. & $\rho_f(\mathrm{Frob}_l)$ has char poly arithmetic
$X^2 - a_l X + l^{k-1}\chi(l)$.

ρ_f may not be irreducible
but it's continuous & odd. ($\det \rho_f(c) = -1$)

Serre asked:

if $\rho : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{F}}_p)$ is cts, odd & irreducible,

then is $\rho \cong \rho_f$ for some f ?

(Yes. if ρ unram @ 2 & $p > 2$
Khare - Wintenberger)

Serre predicted a weight & level & character of f.

Level $=$ cond(ρ) $\qquad$ (by def, prime to p)

$= N(\rho)$

Char & wt mod $p-1$ are uniquely determined
by $\det \rho = \chi \cdot \omega^{k-1}$

ω: mod p cyclo char, $\chi : (\mathbb{Z}/\ldots)^\times \to \overline{\mathbb{F}}_p^\times$

Def'n of k was harder

Fact: if $f \in S_k(\Gamma_1(N), \overline{\mathbb{F}}_p)$ is an eigen form.

$$f = \sum_{n \geq 1} a_n q^n,$$

& if $2 \leq k \leq p+1$.

then we know a lot about $\rho_f \mid D_p$

If $a_p \in \overline{\mathbb{F}}_p$ is non-zero, then $\rho_f \mid D_p = \begin{pmatrix} \omega^{k-1} \times \lambda\left(\frac{\chi(p)}{a_p}\right) & * \\ 0 & \lambda(a_p) \end{pmatrix}$

& in particular $\rho_f \mid I_p = \begin{pmatrix} \omega^{k-1} & * \\ 0 & 1 \end{pmatrix}$.

If $a_p = 0$. then $\rho_f \mid D_p = I(k-1) \otimes \lambda(\sqrt{\chi(p)})$.

i.e. $\rho_f \mid D_p$ is irreducible

det $\rho_f \mid D_p$ is right

& $\rho_f \mid I_p = \begin{pmatrix} \omega_2^{k-1} & 0 \\ 0 & \omega_2^{p(k-1)} \end{pmatrix}$ rk: $p+1 \nmid k-1$ as $2 \leq k \leq p+1$

Given ρ. If $\rho \mid I_p$ is in the above list, one can make a sensible guess for k!

& guess is $\leq p+1$.

Unfortunately $\rho_f \mid D_p$ maybe equal to $\omega^a \oplus \omega^b$ with $0 < a, b < p-1$.

In this case we recall Katz Θ-operator:

if $f = \sum a_n q^n \in S_k(\Gamma_1(N), \overline{\mathbb{F}}_p)$.

then $\exists$ cusp form $\Theta f \in S_{k+p+1}(\Gamma_1(N) : \overline{\mathbb{F}}_p)$.

& q-expansion $(\Theta f) = \sum_{n \geq 1} n a_n q^n$.

$\Theta = q \frac{d}{dq}$ it bumps wts up by $p+1$.

If f is an eigenform, then $\rho_{\Theta f} = \rho_f \otimes \omega$.

[If $X^2 - a_\ell X + \ell^{k-1} \chi(\ell) = (X-\alpha)(X-\beta)$

then $X^2 - \ell \cdot a_\ell X + \ell^{(k+p+1)-1} \cdot \chi(\ell) = (X - \alpha \cdot \ell)(X - \beta \cdot \ell)$

Hence eg if f has wt k & $\rho_f | I_p \sim \begin{pmatrix} \omega^{k-1} & * \\ 0 & 1 \end{pmatrix}$

then $\rho_{\theta f} | I_p = \begin{pmatrix} \omega^k & * \\ 0 & \omega \end{pmatrix}$ & wt$(\theta f) = k + p + 1$.

$\omega^{-1} = \omega^{p-2}$

Given $\bar\rho$, Serre twists by ω^{-1} until he gets something on the list, he then predicts k_0 for this rep'n & set $k = k_0 + n(p+1)$ where n = # times to be twisted

Upshot : Serre predicts $N(\rho)$, $k(\rho) \le p^2$ & $\chi(\rho)$.

Serre's "weak conjecture"

ρ cts ~~=~~ odd irred $\Rightarrow$ ρ modular ($\rho \cong \rho_f$ for some f)

Strong conj : ρ cts odd irred

$\Rightarrow \rho \cong \rho_f$

$f \in S_{k(\rho)}(\Gamma_1(N(\rho)) : \overline{\mathbb{F}}_p)$

char $\chi(\rho)$

$p > 2$

$k(\rho) \not\equiv 1 \bmod p$

unless it's $p+1$

$k(\rho) \not\equiv -1 \bmod p+1$

unless it's p

$f \in S_k(\Gamma_1(N); \overline{\mathbb{F}}_p)$

$a_p = 0$

lift ~~to~~ $F \in S_k(\Gamma_1(N)\, \overline{\mathbb{Q}})$

$|a_p(F)| < 1$

?

2p	3p	4p
2p+1	3p+1	4p+1
	3p+2	4p+2

For $p>2$, weak & strong conj. were proved to be equivalent in 1980/early 90s

Reformulation

If $k \ge 2$

& $f \in S_k(\Gamma_1(N) : \mathbb{C})$

then $\forall \gamma \in \Gamma_1(N)$, the vector $\underline{V}(\gamma) = (V_n)$, $0 \le n \le k-2$.

defined by $V_n = \displaystyle\int_i^{\gamma i} f(z) \cdot z^n \, dz$ is easily checked to be a 1-cocyle $\Gamma \to \mathrm{Sym}^{k-2} \mathbb{C}^2$ (appropriately identified with $\mathbb{C}^{k-1}$)

& Using this one gets a map $\underset{\cap\!\text{I}}{H^1(\Gamma_1(N), \mathrm{Sym}^{k-2}\mathbb{Z}^2)}$

$\underset{\cap\!\text{I}}{S_k(\Gamma_1(N);\mathbb{Z})}$

$$S_k(\Gamma_1(N), \mathbb{C}) \hookrightarrow H^1(\Gamma_1(N), \mathrm{Sym}^{k-2}(\mathbb{C}^2))$$

One can define an action of Hecke ops on RHS to make the map Hecke-equivariant.

A more careful analysis

& one can check that the only systems of Hecke eigenvalues showing up on RHS that don't show up on LHS are associated to Eisenstein series.

Hence if one is chasing around systems of Hecke eigenvalues (an eigenform $f = \sum a_n q^n = q + \dots$ is determined by the $a_n = T_n$-eigenvalue) it doesn't matter which side we work on.

"Reducing right hand side mod p" is tantamount to considering

$$H^1(\Gamma_1(N), \mathrm{Symm}^{k-2}(\overline{\mathbb{F}}_p^2))$$

This guy too has an action of the Hecke op's T_n $\forall n \geq 1$.

Inflation - restriction

G: gp. $K \trianglelefteq G$, normal subgp

M an abelian gp + action of G.

Then $\exists$ long exact seq.

$$0 \to H^1(G/K, M^K) \xrightarrow{\text{inf}} H^1(G, M) \xrightarrow{\text{res}} H^1(K, M)^{G/K} \to H^2(G/K, M^K) \to$$

Let's consider what happens if $G = \Gamma_1(N)$ $(p \nmid N)$

$K = \Gamma_1(N) \cap \Gamma(p)$

$G/K = SL_2(\mathbb{Z}/p\mathbb{Z})$

$M = \mathrm{Sym}^{k-2}(\overline{\mathbb{F}}_p^2)$ stuff here $\supseteq$ e.vals associated to cusp forms wt k level N

We get $0 \to H^1(\underset{\text{finite gp}}{SL_2(\mathbb{Z}/p\mathbb{Z})}, M) \to H^1(\Gamma_1(N), M) \to H^1(\Gamma_1(N) \cap \Gamma(p), M)^{G/K} \to H^2(SL_2(\mathbb{Z}/p\mathbb{Z}), M)$

2 observations:

① The system of eigenvalues showing up in $H^1(SL_2(\mathbb{Z}/p\mathbb{Z}), M)$ are uninteresting.

② We can use Eichler-Shimura again. again to re-interpret $H^1(\Gamma_1(N) \cap \Gamma(p), M)$ in terms of modular forms.

bad reduction at p. (geometrically)

Note finally that

$$H^1(\Gamma_1(N) \cap \Gamma(p), M)^{G/K} = \left(H^1(\Gamma_1(N) \cap \Gamma(p), \overline{\mathbb{F}}_p) \otimes M \right)^{G/K}$$

both rep'n of of G/H. $\quad \mathrm{Sym}^n \overline{\mathbb{F}}_p^2$

$(X \;\; Y) \begin{pmatrix} a & b \\ p & d \end{pmatrix}$

$$= \mathrm{Hom}_{G/K}(M^*, H^1(\Gamma_1(N) \cap \Gamma_0(p), \overline{\mathbb{F}}_p))$$

wt 2 forms level $\Gamma_1(N) \cap \Gamma_0(p)$ where $SL_2(\mathbb{Z}/p\mathbb{Z})$ is acting in a certain way