Lecture ⑥ — Triangulinе properties of the family of Galois rep. on the eigenvarieties (jt work with J. Bellaïche)

(G. Chenevier, April 7)

I) Setting

fix $G_{/\mathbb{Q}}$ definite unitary gp. attached to $E/\mathbb{Q}$ q. im field, rank d.

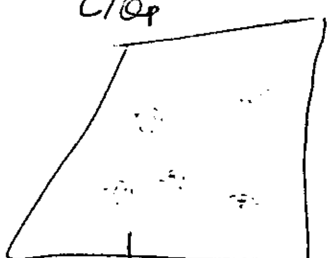
$p = v\bar{v}$ split prime, $G(\mathbb{Q}_p) \xrightarrow{\sim}_v GL_d(\mathbb{Q}_p)$.

$K = \prod_\ell K_\ell$ compact open $\subset G(\mathbb{A}^\infty)$, $K_p = \overset{I}{\text{Iwahori}}$, K_ℓ maximal hyp. $\ell \notin S$ finite.

$\mathcal{H} = \mathcal{A} \otimes_{\mathbb{Z}} \mathbb{Z}[K^S \backslash G(\mathbb{A}^S) / K^S]$

$\uparrow$ $\mathcal{A}$: $[I\,U^{++}I]$ at p

recall — $\psi : \mathcal{H} \to \mathcal{O}(\mathcal{E})^{\leq 1}$ ring hom.



$\mathcal{E}/\mathbb{Q}_p$ rigid space $/\mathbb{Q}_p$

$Z \subset \mathcal{E}$ Zariski-dense subset such that

$z \in Z \iff$ $k(z) = (k_1(z) \leq \dots \leq k_n(z))$, ψ_z is the system of eigenvalues of $\mathcal{H}$ on an eigenform f_z for G, level K

$\downarrow (k_1, \dots, k_d, F_1, \dots, F_d)$

$W \times \mathbb{G}_m^d \leftarrow$ p-refinement of the aut. rep. generated by f_z.

$\uparrow$ weight

$(k_1(z), \dots, k_d(z))$

$(x_1, \dots, x_d) \mapsto x_1^{k_1(z)} \cdots x_d^{k_d(z)}$ if $z \in Z$

$k_1(z) \leq \dots \leq k_d(z)$

$\left(p^{k_1(z)} F_1(z), \dots, p^{k_d(z)} F_d(z)\right)$ are the ordered roots of the Hecke pol., accessible.

$(\varphi_1(z), \dots, \varphi_d(z))$

...ume $\begin{cases} \bullet\ z \in Z,\ \exists \text{ ss galois rep. } \rho_z : G_{E,S} \longrightarrow GL_d(\bar{O}_p) \\ \quad tr\, \rho_z(Frob_v)) = T_v(z) \quad \forall v \notin S. \\ \bullet\ \rho_z \mid G_{E_v} = G_{\mathbb{Q}_p} \text{ is crystalline, HT weights } k_1(z) < k_2(z)+1 < \cdots < k_d(z)+\ldots \end{cases}$

(geometric conventions)

E.g.v. $(.., \varphi_i(z) = p^{k_i(z)} F_i(z), ..)$

NB. Change a bit F, K, such that they give exactly the HTW.

i.e. $K_i(z)$ are the HTW of ρ_z.

At such z Assume moreover $\varphi_i(z)$ are 2 by 2 distincts.

$\rightsquigarrow$ refinement of $D_{crys}(\rho_z \mid G_{\mathbb{Q}_p})$, we will call it $\mathcal{F}_z$.

. T : Global Galois pseudocar., by restriction at $p \Rightarrow T_{\mid G_{\mathbb{Q}_p}} : G_{E,S} \to \mathcal{O}(\mathcal{E})$.

$\Rightarrow$ refined family of Galois representations. (BlahBlah)

(II) Regular crystalline classical points

Fix $z \in Z$, study of the family around z, $A = \mathcal{O}^{rig}_{\mathcal{E},z}$

Assume (REG) $\mathcal{F}_z$ is non critical, regular (local)

$\forall i$, $\varphi_1 \cdots \varphi_i(z)$ is a simple eigenvalue of crys. Frob. on $\Lambda^i \rho_z$.

(IRR) $\Lambda^i \rho_z$ is irred. $\forall i \leq d$. (global).

~~...cofinite length~~,

In particular, $\exists!\ \rho : G_{E,S} \longrightarrow M_d(A)$ trace T.

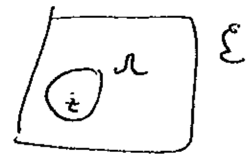
(strictly speaking, as. alg, we kill it by ext. res. field, not important.)

Define $\delta_i : \mathbb{Q}_p^\times \longrightarrow A^\times$, $\delta_i(p) = F_i$

$\delta_{i \mid \Gamma} = K_i^{-1}$

Thm: For each $I \subsetneq A$ cofinite length ideal, $\rho \otimes A/I$ is a trianguline def. of $(\rho_z, \mathcal{F}_z)$ whose parameter is $(\delta_i)_i$.

Sketch pf:



$Z \cap \Omega$ big.

① ρ extends to a neighb. $\rho : G_{E,S} \to GL_d(\mathcal{O}(\Omega))$

② $\rho' = \rho \otimes \delta(\kappa_1)$ has cst. HTW 0, $F_1 : \Omega \to \mathbb{G}_m^{\times}$ is an analytically map. eigenvalue of Frobille fub $Z \cap \Omega$

③ Applying Kisin's construction (conn'd) $\Rightarrow$ (Ω is F_1-small) $\downarrow$ $D^+_{crys}(\rho')^{\varphi = F_1}$ is generically rk 1.

Enough to get $D^+_{cris}\, \rho'_y \neq 0 \quad \forall y \in \Omega$ but not quite immediately to get that.

$$D^+_{cris}\left(\rho' \otimes A/I\right)^{\varphi = F_1} \text{ is free rk 1 over } A/I.$$

In fact yes to do this blow up the problematic ideal and descend the crystalline period after.

$\tilde{\Omega} \to \Omega$ Use that $D_{cris}(\rho'_z)^{\varphi = F_1(z)}$ has dim 1.

④ Apply same construction to $\Lambda^i_{\Gamma}(\delta_1 \cdots \delta_i)$ $\forall i \leq d$, and use.

Prop: ~~Let $(V,\mathcal{F})$ be a non critically refined~~ wt $k_1 < \dots < k_d$, $\varphi_1 \cdots \varphi_d \in L^{\times}$

Let $V_{/L}$ be a crystalline rep. (over L say) equipped with a refinement $\mathcal{F}$ non critical and regular. Let V_A be ~~an~~ a deformation of V and assume there are cont. hom $\delta_i : \mathbb{Q}_p^{\times} \to A^{\times}$ such that $\forall i$

(i) $D_{crys}\left(\Lambda^i V_A (\delta_1 \cdots \delta_i)^{-1}_{|\Gamma}\right)^{\varphi = \delta_1(p) \cdots \delta_i(p)}$ is free rk 1 over A.

(ii) $\delta_{i|\Gamma} \bmod m = \chi^{-t_i}$ $t_i \in \mathbb{Z}$, and $\delta_i(p) = p^{-t_i}\varphi_i$

Then V_A is a trianguline def of $(V,\mathcal{F})$ whose param is $\left(\delta_i\, x^{t_i - k_i}\right)_{i=1\dots d}$.

(where $k_1 < \dots < k_d$ are the HTW of V)

Rk: Using this $R = \Pi^{k_1 k_2} \underset{k}{\simeq} k[X_1 \dots X_d]$, $\to x$ smooth point and assuming $H^1_f(\mathrm{ad}\, \rho_z) = 0$

Reducible points

Let $z \in Z$, any one, $\rho_z = \bar{\rho}_1 \oplus \cdots \oplus \bar{\rho}_r$ (MF as HTW are distinct)

$\mathcal{F}_z = (\varphi_1(z), \ldots, \varphi_d(z))$ induces refinements $\mathcal{F}_{z,i}$ of $\bar{\rho}_i$.

Assume that $\mathcal{F}_{z,i}$ are intervals of $\{1..d\}$, and $\bar{\rho}_i$ ordered, such that:

$$\underbrace{\varphi_1 \cdots \varphi_{d_1}}_{\mathcal{F}_{z,1}} \ \underbrace{\varphi_{d_1+1} \cdots \varphi_{d_2}}_{\mathcal{F}_{z,2}} \ \cdots\cdots \ \underline{\text{etc}}. \quad (\underline{\text{NOT}} \text{ always possible})$$

<u>Define</u> $\sigma \in \mathfrak{S}_d$ associated to this combinatorial datum.

HTW are $k_1(z) < \cdots < k_d(z)$.

$k_{\sigma(1)}(z)$ is the smallest HTW of $\bar{\rho}_1$ $\qquad$ $k_{\sigma(d_1+1)}$ ———— $\bar{\rho}_2$

$\sigma(2)$ ———— 2nd ———— $\bar{\rho}_1$ $\qquad$ <u>etc</u>...

$\vdots$

$k_{\sigma(d_1)}$ ———— d_1^{th} smallest ————

<u>Example</u>

- $\bar{\rho}_i$ have dim 1, hence cryst characters, $r = d$.

  $k_{\sigma(i)}(z)$ is the HTW of $\bar{\rho}_i$, i.e. $v(\varphi_i(z))$.

  $\sigma = id$ iff $\mathcal{F}_z$ is the ordinary ref of ρ_z.

- When σ transitive we say that σ is anti-ordinary.

<u>Assume</u> (Reg$_i$) $\mathcal{F}_{z,i}$ non crit., regular (<u>local</u>)

(Irr) usual assumption $\bigotimes_{i=1}^{d} \Lambda^{a_i} \bar{\rho}_i$, $\forall (a_i)$, $a_i \leq d$.

Global

$T \bmod I_{tot} = \operatorname{tr} \rho_1 + \cdots + \operatorname{tr} \rho_r$, $\quad \rho_i : G_{E,S} \longrightarrow GL_{d_i}(A/I_{tot})$

<u>Theorem</u> Let $I_{tot} \subset J \subset m$. $\forall i$, $\rho_i \otimes A/J$ is a trianguline deformation of $(\bar{\rho}_i, \mathcal{F}_{z,i})$ with explicit parameter, e.g. $i=1$ $\left(p \mapsto (F_1 p^{k_1(t) - k_{\sigma(1)}(1)}, \ldots) \atop r \mapsto (K_{\sigma(1)}^{-1}, \ldots, K_{\sigma(d_1)}^{-1}) \right)$

<u>Moreover</u> $\forall n \in \{1..d\}$, $k_{\sigma(n)} - k_n$ is constant.

Corollary i) Assume that σ is transitive, and that k_m is cst for some
then each k_i is cst. on A/I_{cst}, and A/I_{cst} has dim 0.

ii) If $\forall_i$, $\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\bar{\rho}_i, \bar{\rho}_i(-1)) = 0$, then each ρ_i is crystalline.
moreover

$\Rightarrow$ i) follows from theorem and construction of $\mathcal{E}$.
ii) this follows from lecture 2

Pf theorem basically similar to the irred. case, but extra difficulties coming from the fact that there is no free-module $\supseteq G$ trace is T.
Use lecture 5 to get a good module M_{ij}; A_{ij} structural $\Rightarrow$ become free after a blowup $\Omega \leftarrow \tilde{\Omega}$ here we have a fam. of rep. in the classical sense.
get a family of cryst. periods above.
we prove a lemma showing that $\ell(D_{crys}(M/IM)^{\varphi=F_1})$ is as we want $(= \ell(A/J))$, and some part of M/IM is free $\simeq \rho_A$ enough to get the result.

prop: (Constant weight lemma)

V_L ~~cryst.~~ rep of $G_{\mathbb{Q}_p}$, smallest int. HTW $k \in \mathbb{Z}$, $\lambda \in L^\times$.
Assume that $D_{crys}(V_L)^{\varphi=\lambda}$ ~~[illegible]~~ has dim 1 and its Fil^{k+1} is 0.
Assume that V_A def. such that $D_{crys}(V_A)^{\varphi=\tilde{\lambda}}$, $\tilde{\lambda} \in A^\times$ lifts over λ then k is a constant HTW of V_A.