GALOIS DEFORMATION AND $\mathcal{L}$-INVARIANT

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1. Lecture 2

Let $p > 2$ be a prime, and fix a totally real finite extension $F/\mathbb{Q}$. For **simplicity**, we assume that p splits completely in $F/\mathbb{Q}$. We start with a Galois representation $\rho_F : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to GL_2(W)$ associated to a discrete series Hilbert modular form f (over F) with coefficients in a finite extension $W/\mathbb{Z}_p$ (a DVR). We assume the ordinarity of ρ_F:

$$\rho_F|_{D_\mathfrak{p}} \cong \begin{pmatrix} \epsilon_\mathfrak{p} & * \\ 0 & \alpha_\mathfrak{p} \end{pmatrix} \quad \text{with} \quad \epsilon_\mathfrak{p} \neq \alpha_\mathfrak{p},\ \epsilon_\mathfrak{p}|_{I_\mathfrak{p}} = \mathcal{N}^{k-1} \quad \text{and} \quad \alpha_\mathfrak{p}(I_\mathfrak{p}) = 1$$

on the decomposition group and the inertial group $I_\mathfrak{p} \subset D_\mathfrak{p} \subset \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ for all prime factor $\mathfrak{p}$ of p in F. Here $\mathcal{N}(\sigma) \in \mathbb{Z}_p^\times$ is the p-adic cyclotomic character with $\exp(\frac{2\pi i}{p^n})^\sigma = \exp(\frac{\mathcal{N}(\sigma)2\pi i}{p^n})$ for all $n > 0$ and $k > 1$ is an integer. Again for **simplicity**, we assume that ρ is unramified outside p. Thus for any prime $\mathfrak{l} \nmid p$, writing $f|T(\mathfrak{l}) = a_\mathfrak{l} f$, we have $\mathrm{Tr}(\rho(Frob_\mathfrak{l})) = a_\mathfrak{l} \in W$. Let K be the quotient field of W (so, $K/\mathbb{Q}_p$ is a finite extension).

We consider the **universal** nearly ordinary deformation $\boldsymbol{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to GL_2(R)$ over K with the pro-Artinian local universal K-algebra R. This means that for any Artinian local K-algebra A with maximal ideal $\mathfrak{m}_A$ and any Galois representation $\rho_A : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to GL_2(A)$ such that

(K1) unramified outside p;
(K2) $\rho_A|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/F_\mathfrak{p})} \cong \left(\begin{smallmatrix} * & * \\ 0 & \alpha_{A,\mathfrak{p}} \end{smallmatrix}\right)$ with $\alpha_{A,\mathfrak{p}} \equiv \alpha_\mathfrak{p} \mod \mathfrak{m}_A$;
(K3) $\det(\rho_A) = \det \rho_F$;
(K4) $\rho_A \equiv \rho_F \mod \mathfrak{m}_A$,

there exists a unique K-algebra homomorphism $\varphi : R \to A$ such that $\varphi \circ \boldsymbol{\rho} \cong \rho_A$. Note that $\mathcal{N} : \mathrm{Gal}(F_\mathfrak{p}[\mu_{p^\infty}]/F_\mathfrak{p}) \cong \mathbb{Z}_p^\times$ (by splitting of p in $F/\mathbb{Q}$). Let $\Gamma_\mathfrak{p} = 1 + p\mathbb{Z}_p \subset \mathrm{Gal}(F_\mathfrak{p}[\mu_{p^\infty}]/F_\mathfrak{p})$. Choose a generator $\gamma_\mathfrak{p}$ of $\Gamma_\mathfrak{p}$ and identify $W[[\Gamma_\mathfrak{p}]]$ with $W[[X_\mathfrak{p}]]$ by $\gamma_\mathfrak{p} \leftrightarrow 1 + X_\mathfrak{p}$. Since $\boldsymbol{\rho}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/F_\mathfrak{p})} \cong \left(\begin{smallmatrix} * & * \\ 0 & \boldsymbol{\delta}_\mathfrak{p} \end{smallmatrix}\right)$, $\boldsymbol{\delta}_\mathfrak{p}\alpha_\mathfrak{p}^{-1} : \Gamma_\mathfrak{p} \to R$ induces an algebra structure on R over $W[[X_\mathfrak{p}]]$. Thus R is an algebra over $K[[X_\mathfrak{p}]]_{\mathfrak{p}|p}$. If we write $\varphi : R \to K$ for the morphism with $\varphi \circ \boldsymbol{\rho} \cong \rho_F$, by our construction, $\mathrm{Ker}(\varphi) \supseteq (X_\mathfrak{p})_{\mathfrak{p}|p}$.

Date: May 1, 2006.

The second lecture at Harvard University on 5/1/2006. The author is partially supported by an NSF grant: DMS 0244401 and DMS 0456252.

Here is the theorem we have seen in the first lecture:

Theorem 1.1. *Suppose $R \cong K[[X_{\mathfrak{p}}]]_{\mathfrak{p}|p}$. Then, if $\varphi \circ \boldsymbol{\rho} \cong \rho_F$, for the local Artin symbol $[p, F_{\mathfrak{p}}] = Frob_{\mathfrak{p}}$, we have*

$$\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_F)) = \mathcal{L}(Ad(\rho_F)) = \det\left(\frac{\partial \boldsymbol{\delta}_{\mathfrak{p}}([p, F_{\mathfrak{p}}])}{\partial X_{\mathfrak{p}'}}\right)_{\mathfrak{p},\mathfrak{p}'} \Big|_{X=0} \prod_{\mathfrak{p}} \log_p(\gamma_{\mathfrak{p}}) \alpha_{\mathfrak{p}}([p, F_{\mathfrak{p}}])^{-1}.$$

Greenberg proposed a conjectural formula of the $\mathcal{L}$–invariant for a general p-adic p-ordinary Galois representation V with an exceptional zero. When $V = Ad(\rho_F)$, his definition goes as follows. Under some hypothesis, he found a unique subspace $\mathbb{H} \subset H^1(F, Ad(\rho_F))$ of dimension $e = |\{\mathfrak{p}|p\}|$ represented by cocycles $c : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to Ad(\rho_F)$ such that

(1) c is unramified outside p;
(2) c restricted to $D_{\mathfrak{p}}$ is upper triangular after conjugation for all $\mathfrak{p}|p$.

By the condition (2), $c|_{I_{\mathfrak{p}}}$ modulo upper nilpotent matrices factors through the cyclotomic Galois group $\mathrm{Gal}(\mathbb{Q}_p[\mu_{p^\infty}]/\mathbb{Q}_p)$ because $F_{\mathfrak{p}} = \mathbb{Q}_p$, and hence $c|_{D_{\mathfrak{p}}}$ modulo upper nilpotent matrices becomes unramified everywhere over the cyclotomic $\mathbb{Z}_p$-extension F_∞/F. In other words, the cohomology class $[c]$ is in $\mathrm{Sel}_{F_\infty}(Ad(\rho_F))$ but not in $\mathrm{Sel}_F(Ad(\rho_F))$.

Take a basis $\{c_{\mathfrak{p}}\}_{\mathfrak{p}|p}$ of $\mathbb{H}$ over K. Write

$$c_{\mathfrak{p}}(\sigma) \sim \begin{pmatrix} -a_{\mathfrak{p}}(\sigma) & * \\ 0 & a_{\mathfrak{p}}(\sigma) \end{pmatrix} \quad \text{for } \sigma \in D_{\mathfrak{p}'} \text{ with any } \mathfrak{p}'|p.$$

Then $a_{\mathfrak{p}} : D_{\mathfrak{p}'} \to K$ is a homomorphism. His $\mathcal{L}$-invariant is defined by

$$\mathcal{L}(Ad(\rho_F)) = \det\left((a_{\mathfrak{p}}([p, F_{\mathfrak{p}'}])_{\mathfrak{p},\mathfrak{p}'|p} \left(\log_p(\gamma_{\mathfrak{p}'})^{-1} a_{\mathfrak{p}}([\gamma_{\mathfrak{p}'}, F_{\mathfrak{p}'}])\right)_{\mathfrak{p},\mathfrak{p}'|p}\right)^{-1} \right).$$

The above value is independent of the choice of the basis $\{c_{\mathfrak{p}}\}_{\mathfrak{p}}$. Then assuming the following two conditions:

(1) $\overline{\rho} = (\rho_F \mod \mathfrak{m}_W)$ is absolutely irreducible over $\mathrm{Gal}(\overline{\mathbb{Q}}/F[\mu_p])$;
(ds) $\overline{\rho}^{ss}$ has a non-scalar value over $\mathrm{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}})$ for all prime factors $\mathfrak{p}|p$,

by using a result of Taylor-Wiles and Fujiwara (see Fujiwara's paper: arXiv.math.NT/0602606), we can prove $R \cong K[[X_{\mathfrak{p}}]]$, and the following conjecture for the arithmetic L-function is a theorem except for the nonvanishing $\mathcal{L}(Ad(\rho_F)) \neq 0$ (see [H00] Theorem 6.3 (4)):

Conjecture 1.2 (Greenberg)**.** *Suppose ((ds) and that $\overline{\rho}$ is absolutely irreducible. For $L_p^{arith}(s, Ad(\rho_F)) = \Phi^{arith}(\gamma^s - 1)$, then $L_p^{arith}(s, Ad(\rho_F))$ has zero of order equal to $d = [F : \mathbb{Q}]$ and for the constant $\mathcal{L}(Ad(\rho_F)) \in K^\times$ specified by the determinant as in the theorem, we have*

$$\lim_{s \to 1} \frac{L_p^{arith}(s, Ad(\rho_F))}{(s-1)^d} = \mathcal{L}(Ad(\rho_F)) \mathcal{E}^+(Ad(\rho_F)) \big|\big|\mathrm{Sel}_{\mathbb{Q}}(\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_F)^*)\big|\big|_p^{-1/[K:\mathbb{Q}_p]}$$

up to units.

In the following section, we describe the Selmer group and how to define $\mathbb{H}$.

1.1. **Selmer Groups.** We recall Greenberg's definition of Selmer groups. Write $F^{(p)}/F$ for the maximal extension unramified outside p and ∞. Put $\mathfrak{G} = \mathrm{Gal}(F^{(p)}/F)$ and $\mathfrak{G}_M = \mathrm{Gal}(F^{(p)}/M)$. Let $V = Ad(\rho_F)$ with a continuous action of $\mathfrak{G}$. We fix a W-lattice T in V stable under $\mathfrak{G}$.

Write $D = D_\mathfrak{p} \subset \mathfrak{G}$ for the decomposition group of each prime factor $\mathfrak{p}|p$. Choosing a basis of ρ_F so that $\rho_F|_D$ is upper triangular. We have a 3-step filtration:

$$V \supset \mathcal{F}^-_\mathfrak{p} V \supset \mathcal{F}^+_\mathfrak{p} V \supset \{0\}, \tag{ord}$$

where taking a basis so that $\rho_F|_D$ is upper triangular, $\mathcal{F}^-_\mathfrak{p} V$ is made up of upper triangular matrices and $\mathcal{F}^+_\mathfrak{p} V$ is made up of upper nilpotent matrices, and on $\mathcal{F}^-_\mathfrak{p} V/\mathcal{F}^+_\mathfrak{p} V$, D acts trivially (getting eigenvalue 1 for $Frob_\mathfrak{p}$). Since V is self-dual, its dual $V^*(1) = \mathrm{Hom}_K(V, K) \otimes \mathcal{N}$ again satisfies (ord).

Let M/F be a subfield of $F^{(p)}$, and put $\mathfrak{G}_M = \mathrm{Gal}(F^{(p)}/M)$. We write $\mathfrak{p}$ for a prime of M over p and $\mathfrak{q}$ for general primes of M. We put

$$L_\mathfrak{p}(V) = \mathrm{Ker}(\mathrm{Res} : H^1(M_\mathfrak{p}, V) \to H^1(I_\mathfrak{p}, \frac{V}{\mathcal{F}^+_\mathfrak{p}(V)})).$$

Then for a $\mathfrak{G}_M$-stable W-lattice T of V, we define for the image $L_\mathfrak{p}(V/T)$ of $L_\mathfrak{p}(V)$ in $H^1(M_\mathfrak{p}, V/T)$

$$\mathrm{Sel}_M(A) = \mathrm{Ker}(H^1(\mathfrak{G}_M, A) \to \prod_\mathfrak{p} \frac{H^1(M_\mathfrak{p}, A)}{L_\mathfrak{p}(A)})) \quad \text{for } A = V, V/T. \tag{1.1}$$

The classical Selmer group of V is given by $\mathrm{Sel}_M(V/T)$, equipped with discrete topology. Write F_∞ for the cyclotomic $\mathbb{Z}_p$–extension of F. We define "$-$" Selmer group replacing $L_\mathfrak{p}(A)$ by

$$L^-_\mathfrak{p}(V) = \mathrm{Ker}(\mathrm{Res} : H^1(M_\mathfrak{p}, V) \to H^1(I_\mathfrak{p}, \frac{V}{\mathcal{F}^-_\mathfrak{p}(V)})).$$

Lemma 1.3. *Suppose* $R \cong K[[X_\mathfrak{p}]]_{\mathfrak{p}|p}$. *Then* $\mathrm{Sel}^-_F(V) \cong \mathrm{Hom}_K(\mathfrak{m}_R/\mathfrak{m}_R^2, K)$ *and* $\mathrm{Sel}_F(V) = 0$.

Proof. We consider the space $Der_K(R, K)$ of continuous K-derivations. Let $K[\varepsilon] = K[t]/(t^2)$ for the dual number $\varepsilon = (t \mod t^2)$. Then writing K-algebra homomorphism $\phi : R \to K[\varepsilon]$ as $\phi(r) = \phi_0(r) + \phi_1(r)\varepsilon$ and sending ϕ to $\phi_1 \in Der_K(R, K)$, we have $\mathrm{Hom}_{K\text{-alg}}(R, K[\varepsilon]) \cong Der_K(R, K) = \mathrm{Hom}_K(\mathfrak{m}_R/\mathfrak{m}_R^2, K)$. By the universality of $(R, \boldsymbol{\rho})$, we have

$$\mathrm{Hom}_{K\text{-alg}}(R, K[\varepsilon]) \cong \frac{\{\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to GL_2(K[\varepsilon])|\rho \text{ satisfies the condtions (K1–4)}\}}{\cong}.$$

Pick ρ as above. Write $\rho(\sigma) = \rho_0(\sigma) + \rho_1(\sigma)\varepsilon$. Then $c_\rho = \rho_1\rho_F^{-1}$ can be easily checked to be a 1-cocycle having values in $M_2(K) \supset V$. Since $\det(\rho) = \det(\rho_F) \Rightarrow \mathrm{Tr}(c_\rho) = 0$,

c_ρ has values in V. By the reducibility condition (K2), $[c_\rho] \in \mathrm{Sel}^-_F(V)$. We see easily that $\rho \cong \rho' \Leftrightarrow [c_\rho] = [c_{\rho'}]$. We can reverse the above argument starting a cocycle c giving an element of $\mathrm{Sel}^-_F(V)$ to construct a deformation ρ_c with values in $K[\varepsilon]$. Thus we have

$$\frac{\{\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to GL_2(K[\varepsilon])|\rho \text{ satisfies the condtions (K1–4)}\}}{\cong} \cong \mathrm{Sel}^-_F(V).$$

Since the algebra structure of R over $W[[X_\mathfrak{p}]]_{\mathfrak{p}|p}$ is given by $\boldsymbol{\delta}_\mathfrak{p}\alpha_\mathfrak{p}^{-1}$, the K-derivation $\delta : R \to K$ corresponding to a $K[\varepsilon]$-deformation ρ is a $W[[X_\mathfrak{p}]]$-derivation if and only if $\rho_1|_{\mathrm{Gal}(\overline{F}_\mathfrak{p}/F_\mathfrak{p})} \sim \left(\begin{smallmatrix} * & * \\ 0 & 0 \end{smallmatrix}\right)$, which is equivalent to $[c_\rho] \in \mathrm{Sel}_F(V)$, because we already knew that $\mathrm{Tr}(c_\rho) = 0$. Thus we have $\mathrm{Sel}_F(V) \cong Der_{W[[X_\mathfrak{p}]]}(R, K) = 0$. □

We also have

Lemma 1.4.

$$\text{(V)} \qquad \mathrm{Sel}_F(V) = 0 \Rightarrow H^1(\mathfrak{G}, V) \cong \prod_\mathfrak{p} \frac{H^1(F_\mathfrak{p}, V)}{L_\mathfrak{p}(V)}.$$

Indeed, by the Poitou-Tate exact sequence, the following sequence is exact:

$$\mathrm{Sel}_F(V) \to H^1(\mathfrak{G}_M, V) \to \prod_\mathfrak{p} \frac{H^1(F_\mathfrak{p}, V)}{L_\mathfrak{p}(V)} \to \mathrm{Sel}_F(V^*(1))^*.$$

It is an old theorem of Greenberg that $\dim \mathrm{Sel}_F(V) = \dim \mathrm{Sel}_F(V^*(1))^*$ (see [G] Proposition 2); so, we have the assertion (V). □

2. Greenberg's $\mathcal{L}$–invariant

Here is Greenberg's definition of $\mathcal{L}(V)$: The long exact sequence of $\mathcal{F}^-_\mathfrak{p}V/\mathcal{F}^+_\mathfrak{p}V \hookrightarrow V/\mathcal{F}^+_\mathfrak{p}V \twoheadrightarrow V/\mathcal{F}^-_\mathfrak{p}V$ gives a homomorphism, noting $F_\mathfrak{p} = \mathbb{Q}_p$,

$$H^1(F_\mathfrak{p}, \mathcal{F}^-_\mathfrak{p}V/\mathcal{F}^+_\mathfrak{p}V) = \mathrm{Hom}(\mathfrak{G}^{ab}_{\mathbb{Q}_p}, \mathcal{F}^-_\mathfrak{p}V/\mathcal{F}^+_\mathfrak{p}V) \xrightarrow{\iota_\mathfrak{p}} H^1(F_\mathfrak{p}, V)/L_\mathfrak{p}(V).$$

Note that

$$\mathrm{Hom}(G^{ab}_{\mathbb{Q}_p}, \mathcal{F}^-_\mathfrak{p}V/\mathcal{F}^+_\mathfrak{p}V) \cong (\mathcal{F}^-_\mathfrak{p}V/\mathcal{F}^+_\mathfrak{p}V)^2 \cong K^2$$

canonically by $\phi \mapsto (\frac{\phi([\gamma, F_\mathfrak{p}])}{\log_p(\gamma)}, \phi([p, F_\mathfrak{p}]))$. Here $[x, F_\mathfrak{p}] = [x, \mathbb{Q}_p]$ is the local Artin symbol (suitably normalized). Since

$$L_p(\mathcal{F}^-_\mathfrak{p}V/\mathcal{F}^+_\mathfrak{p}V) = \mathrm{Ker}(H^1(F_\mathfrak{p}, \mathcal{F}^-_\mathfrak{p}V/\mathcal{F}^+_\mathfrak{p}V) \xrightarrow{\mathrm{Res}} H^1(I_\mathfrak{p}, \mathcal{F}^-_\mathfrak{p}V/\mathcal{F}^+_\mathfrak{p}V)),$$

the image of $\iota_\mathfrak{p}$ is isomorphic to $\mathcal{F}^-_\mathfrak{p}V/\mathcal{F}^+_\mathfrak{p}V \cong K$. By (V), we have a unique subspace $\mathbb{H}$ of $H^1(\mathfrak{G}, V)$ projecting down onto

$$\prod_\mathfrak{p} \mathrm{Im}(\iota_\mathfrak{p}) \hookrightarrow \prod_\mathfrak{p} \frac{H^1(F_\mathfrak{p}, V)}{L_\mathfrak{p}(V)}.$$

Then by the restriction, $\mathbb{H}$ gives rise to a subspace L of

$$\prod_{\mathfrak{p}} \mathrm{Hom}(G^{ab}_{F_{\mathfrak{p}}}, \mathcal{F}^-_{\mathfrak{p}}V/\mathcal{F}^+_{\mathfrak{p}}V) \cong \prod_{\mathfrak{p}} (\mathcal{F}^-_{\mathfrak{p}}V/\mathcal{F}^+_{\mathfrak{p}}V)^2$$

isomorphic to $\prod_{\mathfrak{p}}(\mathcal{F}^-_{\mathfrak{p}}V/\mathcal{F}^+_{\mathfrak{p}}V)$. If a cocycle c representing an element in $\mathbb{H}$ is unramified, it gives rise to an element in $\mathrm{Sel}_F(V)$. By the vanishing of $\mathrm{Sel}_F(V)$ (Lemma 1.3), this implies $c = 0$; so, the projection of L to the first factor $\prod_{\mathfrak{p}}(\mathcal{F}^-_{\mathfrak{p}}V/\mathcal{F}^+_{\mathfrak{p}}V)$ (via $\phi \mapsto (\phi([\gamma, F_{\mathfrak{p}}])/\log_p(\gamma))_{\mathfrak{p}})$ is surjective. Thus this subspace L is a graph of a K–linear map $\mathcal{L} : \prod_{\mathfrak{p}} \mathcal{F}^-_{\mathfrak{p}}V/\mathcal{F}^+_{\mathfrak{p}}V \to \prod_{\mathfrak{p}} \mathcal{F}^-_{\mathfrak{p}}V/\mathcal{F}^+_{\mathfrak{p}}V$. We then define $\mathcal{L}(V) = \det(\mathcal{L}) \in K$.

Let $\boldsymbol{\rho} : \mathfrak{G}_F \to GL_2(R)$ be the universal nearly ordinary deformation with $\boldsymbol{\rho}\big|_D = \begin{pmatrix} * & * \\ 0 & \boldsymbol{\delta} \end{pmatrix}$. Then $c_{\mathfrak{p}} = \frac{\partial \boldsymbol{\rho}}{\partial X_{\mathfrak{p}}}|_{X=0}\rho_F^{-1}$ is a 1-cocycle (by the argument proving Lemma 1.3) giving rise to a class of $\mathbb{H}$. By Lemma 1.3, $\mathbb{H} = \mathrm{Sel}^-_F(V)$, and $\{c_{\mathfrak{p}}\}_{\mathfrak{p}}$ gives a basis of $\mathbb{H}$ over K. We have $\boldsymbol{\delta}([u, F_{\mathfrak{p}}]) = (1 + X_{\mathfrak{p}})^{\log_p(u)/\log_p(\gamma)}$ for $u \in O^\times_{\mathfrak{p}} = \mathbb{Z}^\times_p$. Writing

$$c_{\mathfrak{p}}(\sigma) = \begin{pmatrix} -a_{\mathfrak{p}}(\sigma) & * \\ 0 & a_{\mathfrak{p}}(\sigma) \end{pmatrix} \rho_F(\sigma)^{-1},$$

we have $a_{\mathfrak{p}} = \boldsymbol{\delta}^{-1}\frac{d\boldsymbol{\delta}}{dX_{\mathfrak{p}}}|_{X=0}$, and from this we get the desired formula of $\mathcal{L}(Ad(\rho_F))$.

If one restricts $c \in \mathbb{H}$ to $\mathfrak{G}_\infty = \mathrm{Gal}(F^{(p)}/F_\infty)$, its ramification is exhausted by $\Gamma = \mathrm{Gal}(F_\infty/F)$ (because $F_{\mathfrak{p}} = \mathbb{Q}_p$) giving rise to a class $[c] \in \mathrm{Sel}_{F_\infty}(V)$. The kernel of the restriction map: $H^1(\mathfrak{G}, V) \to H^1(\mathfrak{G}_\infty, V)$ is given by $H^1(\Gamma, H^0(\mathfrak{G}_\infty, V)) = 0$ because $H^0(\mathfrak{G}_\infty, V) = 0$. Thus the image of $\mathcal{H}$ in $\mathrm{Sel}_{F_\infty}(V/T)$ gives rise to the order d exceptional zero of $L^{arith}(s, Ad(\rho_F))$ at $s = 1$. We have proved

Proposition 2.1. *For the number of prime factors $d = [F : \mathbb{Q}]$ of p in F, we have*

$$\mathrm{ord}_{s=1} L^{arith}_p(s, Ad(\rho_F)) \geq d.$$

References

[G] R. Greenberg, Trivial zeros of p–adic L–functions, Contemporary Math. **165** (1994), 149–174

[GS] R. Greenberg and G. Stevens, p-adic L–functions and p–adic periods of modular forms, Inventiones Math. **111** (1993), 407–447

[H00] H. Hida, Adjoint Selmer groups as Iwasawa modules, Israel Journal of Math. **120** (2000), 361–427

[MFG] H. Hida, *Modular Forms and Galois Cohomology*, Cambridge Studies in Advanced Mathematics **69**, 2000, Cambridge University Press