ELLIPTIC CURVES WITH MULTIPLICATIVE REDUCTION

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1. Lecture 3

Let p be an odd prime. Order the prime factors of p as $\mathfrak{p}_1, \dots, \mathfrak{p}_g$. In this lecture, we describe the computation of the $\mathcal{L}$-invariant of $Ad(T_pE)$ for a modular elliptic curve $E_{/F}$ with split multiplicative reduction at $\mathfrak{p}_j|p > 2$ for $j = 1, 2, \dots, k$ and ordinary good reduction at $\mathfrak{p}_j|p$ for $j > k$.

Theorem 1.1. *Assume that $R \cong \mathbb{Q}_p[[X_\mathfrak{p}]]_{\mathfrak{p}|p}$. Suppose that the Hilbert-modular elliptic curve E has split multiplicative reductionat $\mathfrak{p}_j$ for $j = 1, 2, \dots, k$ $(k \le g)$ with Tate period q_j at $\mathfrak{p}_j$ for $j \le k$ and has ordinary good reduction at $\mathfrak{p}_i$ with $i > k$. Then for the local Artin symbol $[p, F_\mathfrak{p}] = Frob_\mathfrak{p}$ and the norm $Q_j = N_{F_{\mathfrak{p}_j}/\mathbb{Q}_p}(q_j)$, we have for $\rho_E = T_pE$*

$$\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_E)) = \left(\prod_{j=1}^{k} \frac{\log_p(Q_j)}{\mathrm{ord}_p(Q_j)}\right) \cdot \det\left(\frac{\partial \boldsymbol{\delta}_\mathfrak{p}([p, F_i])}{\partial X_j}\right)_{i>k, j>k} \Big|_{X=0} \prod_{i>k} \frac{\log_p(\gamma_{\mathfrak{p}_i})}{\alpha_\mathfrak{p}([p, F_i])},$$

where $\gamma_\mathfrak{p}$ is the generator of the p-profinite part $\Gamma_\mathfrak{p}$ of $\mathcal{N}(\mathrm{Gal}(F_\mathfrak{p}[\mu_{p^\infty}]/F_\mathfrak{p}))$ by which we identify the group algebra $W[[\Gamma_\mathfrak{p}]]$ with $W[[X_\mathfrak{p}]]$.

In the proof, for simplicity, as before, we assume that p is completely split in $F/\mathbb{Q}$. Also, again for simplicity, in the following proof, we assume E has good reduction outside p and $k = 1$. We put $\Gamma_F = \prod_\mathfrak{p} \Gamma_\mathfrak{p}$.

1.1. **Hecke algebras for quaternion algebras.** We make some preparation for the proof, gathering known facts. We assume that $F \neq \mathbb{Q}$ (otherwise the theorem is known by Greenberg-Stevens). For simplicity, p splits completely in $F/\mathbb{Q}$. Take first a quaternion algebra $B_{0/F}$ central over F unramified everywhere such that $B_0 \otimes_\mathbb{Q} \mathbb{R} \cong M_2(\mathbb{R})^r \times \mathbb{H}^{d-r}$ with $0 \le r \le 1$ (so $r \equiv d \mod 2$). Then we consider the automorphic variety (either a Shimura curve $(r = 1)$ or a 0-dimensional point set $(r = 0)$) given by

$$X_{11}(p^n) = B_0^\times \backslash B_{0,\mathbb{A}}^\times / S_{11}(p^n) Z_\mathbb{A} C_\infty,$$

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where $Z_{\mathbb{A}} \cong F_{\mathbb{A}}^{\times}$ is the center of $B_{\mathbb{A}}^{\times}$, C_∞ is a maximal compact subgroup of the identity component of $B_{0,\infty}^{\times}$ and identifying $B_{0,\mathfrak{l}}^{(\infty)} = B_0 \otimes_{\mathbb{Q}} F_{\mathfrak{l}}$ with $M_2(F_{\mathfrak{l}})$ for all primes $\mathfrak{l}$,

$$S_{11}(p^n) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\widehat{O}) \middle| \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \mod p^n \right\}$$

for $\widehat{O} = \prod_{\mathfrak{l}} O_{\mathfrak{l}}$. Consider $M_n \cong H_r(X_{11}(p^n), \mathbb{Z}_p)$ which is the Pontryagin dual of $H^r(X_{11}(p^n), \mathbb{Q}_p/\mathbb{Z}_p)$ which is a finite rank free $\mathbb{Z}_p$-module with Hecke operator action of $T(\mathfrak{n})$ for all prime ideals outside p and $U(p_{\mathfrak{p}}^n) = U(p_{\mathfrak{p}})^n$ and the diamond operator action $\langle z \rangle$ coming from $\left(\begin{smallmatrix} z & 0 \\ 0 & 1 \end{smallmatrix}\right)$ for $z \in O_p$. Let $e = \lim_{n\to\infty} U(p)^{n!}$ as an operator acting on M_n ($U(p) = \prod_{\mathfrak{p}} U(p_{\mathfrak{p}})$). Let M_n^{ord} be the direct summand eM_n. We have natural trace map $M_m \twoheadrightarrow M_n$ for $m > n$ compatible with all Hecke operators and all diamond operators. By the diamond operator action, $M_\infty^{ord} = \varprojlim_n M_n^{ord}$ naturally become a $W[[\Gamma_F]]$-module. Here is an old theorem of mine:

Theorem 1.2. *The $W[[\Gamma_F]]$-module M_∞^{ord} is free of finite rank over $W[[\Gamma_F]]$.*

Let $\mathbf{h}$ be the $W[[\Gamma_F]]$-algebra generated over $W[[\Gamma_F]]$ by $T(\mathfrak{n})$ for all $\mathfrak{n}$ prime to p and all $U(\mathfrak{p})$. Then we have

Corollary 1.3. $\mathbf{h}$ *is torsion free of finite type over $W[[\Gamma_F]]$ with $\mathbf{h}_F/(X_{\mathfrak{p}})_{\mathfrak{p}|p}\mathbf{h}_F$ pseudo isomorphic to the Hecke algebra of $H_r(X_{11}(p), W)$.*

Actually if $p \geq 5$, $\mathbf{h}$ is known to be free over $W[[\Gamma_F]]$ and the pseudo isomorphism as above is actually an isomorphism.

Let $\mathbb{T}$ be the local ring of the universal nearly ordinary Hecke algebra $\mathbf{h}$ acting non-trivially on the Hecke eigenform associated to E. Let $P \in \mathrm{Spf}(\mathbb{T})(\mathbb{Q}_p)$ corresponding to ρ_E, that is, $\rho_{\mathbb{T}} \mod P \sim \rho_E$. Let $\widehat{\mathbb{T}}_P = \varprojlim_n \mathbb{T}_P/P^n\mathbb{T}_P$ for the localization $\mathbb{T}_P$. Since $\rho_E = T_pE \otimes \mathbb{Q}_p$ is absolutely irreducible, by the technique of pseudo representation, we can construct the modular deformation $\rho_{\mathbb{T}} : \mathfrak{G} \to GL_2(\widehat{\mathbb{T}}_P)$ which satisfies (K1–4); in particular, $\det \rho_{\mathbb{T}} = \mathcal{N}$, because the central character is trivial. Since E is modular over F, we have the surjective $\mathbb{Q}_p$-algebra homomorphism $R \to \widehat{\mathbb{T}}_P$ for the localization-completion $\widehat{\mathbb{T}}_P$. Since $\widehat{\mathbb{T}}_P$ is integral and of dimension d, we have

Corollary 1.4. *If $R \cong K[[X_{\mathfrak{p}}]]_{\mathfrak{p}|p}$, then $R \cong \widehat{\mathbb{T}}_P$.*

The isomorphism $R \cong K[[X_{\mathfrak{p}}]]_{\mathfrak{p}|p}$ is proven by showing $R \cong \widehat{\mathbb{T}}_P$ first (see Appendix).

Take a quaternion algebra $B_{1/F}$ such that $B_1 \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R})^q \times \mathbb{H}^{d-q}$ with $q \leq 1$ and B is ramified only at $\mathfrak{p}_1$ (among finite places). Then at $\mathfrak{p}_1$, we have a unique maximal order R_1 in $B_{\mathfrak{p}_1}$. Then we define $U_{11}(p^n)$ to be the product of $S_{11}(p^n)^{(\mathfrak{p}_1)}$ and $R_1^{\times}$ and define

$$Y_{11}(p^n) = B_1^{\times} \backslash B_{1,\mathbb{A}}^{\times} / U_{11}(p^n) Z_{\mathbb{A}} C_\infty.$$

Then we define $e_1 = \lim_{n\to\infty} U(p^{(\mathfrak{p}_1)})^{n!}$ acting on the dual $N_n = H_q(Y_{11}(p^n), \mathbb{Z}_p)$ of the cohomology group $H^q(Y_{11}(p^n), \mathbb{Q}_p/\mathbb{Z}_p)$. Let $\Gamma_1 = \prod_{\mathfrak{p} \neq \mathfrak{p}_1} \Gamma_{\mathfrak{p}}$. We go through all the above process and define $\mathbf{h}_1 \subset \mathrm{End}_{W[[\Gamma_1]]}(\varprojlim_n e_1 N_n)$. Since ρ_E (or corresponding automorphic representation π_E) is Steinberg at $\mathfrak{p}_1$, by the Jacquet-Langlands correspondence, we have a Hecke eigenvector f_1 in $H^q(Y_{11}(p), \mathbb{Z}_p)$ giving rise to E. Then we define $\mathbb{T}_1$ to be the local ring of $\mathbf{h}_1$ acting nontrivially on f_1. Let $P_1 \in \mathrm{Spf}(\mathbb{T}_1)(W)$ be the point associated to ρ_E. We then have a deformation $\rho_{\mathbb{T}_1} : \mathfrak{G} \to GL_2(\widehat{\mathbb{T}}_{1,P})$ of ρ_E. Since the central character is trivial, we have $\det \rho_{\mathbb{T}_1} = \mathcal{N}$.

Theorem 1.5. *We have*

(1) $\mathbf{h}_1$ *is torsion-free of finite rank over* $W[[\Gamma_1]]$*, and* $\widehat{\mathbb{T}}_{1,P_1} \cong K[[X_{\mathfrak{p}_2}, \dots, X_{\mathfrak{p}_d}]]$*;*
(2) $\rho_{\mathbb{T}_1}$ *restricted to* $\mathrm{Gal}(\overline{F}_{\mathfrak{p}_1}/F_{\mathfrak{p}_1})$ *is isomorphic to* $\left(\begin{smallmatrix} \varepsilon\mathcal{N} & * \\ 0 & \varepsilon \end{smallmatrix}\right)$*, where* $\varepsilon = \pm 1$ *is the eigenvalue of* $Frob_{\mathfrak{p}_1}$ *on the étale quotient of* T_pE*;*
(3) *There is a surjective algebra homomorphism* $\mathbb{T}/X_{\mathfrak{p}_1}\mathbb{T} \twoheadrightarrow \mathbb{T}_1$ *inducing an isomorphism* $\widehat{\mathbb{T}}_P/X_{\mathfrak{p}_1}\widehat{\mathbb{T}}_P \cong \widehat{\mathbb{T}}_{1,P_1}$*;*
(4) *There is a surjective algebra homomorphism* $\mathbb{T}/(U(\mathfrak{p}_1) - \varepsilon)\mathbb{T} \twoheadrightarrow \mathbb{T}_1$ *sending* $T(\mathfrak{n})$ *to* $T(\mathfrak{n})$*, where* $U(\mathfrak{p}_1) = U(p_{\mathfrak{p}_1})$*.*

Here is a sketch of proof. The first assertion follows from construction; in other words, it can be proven by the same way as the proof of Corollary 1.3. By the Jacquet-Langlands correspondence, $\mathbb{T}$ covers $\mathbb{T}_1$. Any automorphic representation π corresponding to a point of $\mathrm{Spf}(\mathbb{T}_1)(\overline{\mathbb{Q}}_p)$ is Steinberg at $\mathfrak{p}_1$ because B_1 ramifies at $\mathfrak{p}_1$. Since points corresponding classical automorphic representation is Zariski dense in $\mathrm{Spf}(\mathbb{T}_1)$, the Galois representation has to have the form as in (2). Thus the eigenvalue of $U(\mathfrak{p}_1)$ of π is ± 1 and the corresponding Galois representation has the form as in (2). The assertion (1) implies (3). By (2), $U(\mathfrak{p}_1)$ is either ± 1. Since $U(\mathfrak{p}_1)$ is a formal function on the connected $\mathrm{Spf}(\mathbb{T}_1)$, $U(\mathfrak{p}_1) = \varepsilon$ is a constant, which implies (4). □

1.2. **Proof of Theorem 1.1.** Write for simplicity, $X_j := X_{\mathfrak{p}_j}$, $F_j = F_{\mathfrak{p}_j}$ and $p_j = p_{\mathfrak{p}_j}$. By (3) and (4) of Theorem 1.5, $U(\mathfrak{p}_1) \equiv \varepsilon \mod X_1$ is a constant independent of $X_j := X_{\mathfrak{p}_j}$ for all $j \geq 2$. Thus $\frac{\partial U(\mathfrak{p}_1)}{\partial X_j}|_{X_1=0} = 0$ for all $j \geq 2$. Thus

$$\det\left(\frac{\partial U(\mathfrak{p}_i)}{\partial X_j}\right)\Big|_{X=0} = \frac{\partial U(\mathfrak{p}_1)}{\partial X_1}\Big|_{X_1=0} \times \det\left(\frac{\partial U(\mathfrak{p}_i)}{\partial X_j}\right)_{i\geq 2, j\geq 2}\Big|_{X=0}.$$

Since $\boldsymbol{\delta}_{\mathfrak{p}_i}([p, F_i]) = U(\mathfrak{p}_i)$, we get from the formula we stated in the first lecture:

$$\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_E)) = \det\left(\frac{\partial \boldsymbol{\delta}_{\mathfrak{p}}([p, F_{\mathfrak{p}}])}{\partial X_{\mathfrak{p}'}}\right)_{\mathfrak{p},\mathfrak{p}'}\Big|_{X=0} \prod_{\mathfrak{p}} \log_p(\gamma_{\mathfrak{p}})\alpha_{\mathfrak{p}}([p, F_{\mathfrak{p}}])^{-1},$$

the following new formula:

$$(1.1)\quad \mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_E)) = \frac{\partial \boldsymbol{\delta}_{\mathfrak{p}}([p,F_1])}{\partial X_1}\Big|_{X_1=0} \log_p(\gamma_{\mathfrak{p}_1})\alpha_{\mathfrak{p}_1}([p,F_1])^{-1} \times \det\left(\frac{\partial \boldsymbol{\delta}_{\mathfrak{p}}([p,F_i])}{\partial X_j}\right)_{i\geq 2, j\geq 2}\Big|_{X=0} \prod_{j\geq 2} \log_p(\gamma_{\mathfrak{p}_j})\alpha_{\mathfrak{p}}([p,F_j])^{-1}.$$

Thus the result follows from the following result of Greenberg-Stevens:

Lemma 1.6. *Let us write $\gamma = \gamma_{\mathfrak{p}_1}$. We have*

$$\frac{\partial \boldsymbol{\delta}_1([p,F_1])}{\partial X_1}\Big|_{X_1=0} \log_p(\gamma)\alpha_{\mathfrak{p}_1}([p,F_1])^{-1} = \frac{\log_p(q_1)}{\mathrm{ord}_p(q_1)}$$

for $\boldsymbol{\delta}_1 = \boldsymbol{\delta}_{\mathfrak{p}_1}$.

Proof. Since $\alpha_{\mathfrak{p}_1}([p,F_1]) = 1$ (split multiplicative reduction), we can forget about this factor. Since the matrix of the linear operator $\mathcal{L} : \prod_{\mathfrak{p}} \mathcal{F}_{\mathfrak{p}}^- V/\mathcal{F}_{\mathfrak{p}}^+ V \to \prod_{\mathfrak{p}} \mathcal{F}_{\mathfrak{p}}^- V/\mathcal{F}_{\mathfrak{p}}^+ V$ induces $\mathcal{L}_1 : \mathcal{F}_{\mathfrak{p}_1}^- V/\mathcal{F}_{\mathfrak{p}_1}^+ V \to \mathcal{F}_{\mathfrak{p}_1}^- V/\mathcal{F}_{\mathfrak{p}_1}^+ V$ by our diagonalization of its matrix. This $\mathcal{L}_1$ comes from the subspace

$$L_1 \subset \mathrm{Hom}(D_1^{ab}, \mathcal{F}_{\mathfrak{p}_1}^- V/\mathcal{F}_{\mathfrak{p}_1}^+ V) \cong \mathrm{Hom}(D_1^{ab}, \mathbb{Q}_p)$$

for $D_1 = \mathrm{Gal}(\overline{\mathbb{Q}}_p/F_1)$ has a generator $\phi_0 = \boldsymbol{\delta}_1^{-1} \frac{\partial \boldsymbol{\delta}_1}{\partial X_1}\Big|_{X_1=0} : D_1^{ab} \to \mathbb{Q}_p$. Thus by definition

$$\frac{\partial \boldsymbol{\delta}_1([p,F_1])}{\partial X_1}\Big|_{X_1=0} \log_p(\gamma) = \log_p(\gamma)\frac{\phi_0([p,F_1])}{\phi_0([\gamma,F_1])}.$$

Let $\rho_E = T_pE \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, $\widetilde{\rho}_E = (\boldsymbol{\rho} \mod (X_1^2, X_2, \ldots, X_d))$, and write $\widetilde{\mathbb{Q}_p} = \mathbb{Q}_p[X_1]/(X_1^2)$. The character $(\boldsymbol{\delta}_1 \mod X_1^2)$ is an infinitesimal deformation of the trivial character fitting into the following commutative diagram of D_1-modules:

$$\begin{array}{ccccc} \widetilde{\mathbb{Q}_p}(\boldsymbol{\epsilon}_1) & \xrightarrow{\hookrightarrow} & \widetilde{\rho}_E & \xrightarrow{\twoheadrightarrow} & \widetilde{\mathbb{Q}_p}(\boldsymbol{\delta}_1) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{Q}_p(1) & \longrightarrow & \rho_E & \longrightarrow & \mathbb{Q}_p. \end{array}$$

Twist this diagram by $\boldsymbol{\epsilon}_1^{-1}\mathcal{N} = \boldsymbol{\delta}_1$, getting a new diagram

$$\begin{array}{ccccc} \widetilde{\mathbb{Q}_p}(1) & \xrightarrow{\hookrightarrow} & \widetilde{\rho}_E & \xrightarrow{\twoheadrightarrow} & \widetilde{\mathbb{Q}_p}(\boldsymbol{\delta}_1^2) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{Q}_p(1) & \longrightarrow & \rho_E & \longrightarrow & \mathbb{Q}_p. \end{array}$$

Once this type of diagram is obtained (with leftmost column given by $\widetilde{\mathbb{Q}_p}(1) \twoheadrightarrow \mathbb{Q}_p(1)$), by a general result of Greenberg-Stevens in such a situation (see [GS1] (2.3.4) and

Theorem 1.14 in the text), we get

$$\frac{\partial \boldsymbol{\delta}_1^2}{\partial X_1}([q_1, \mathbb{Q}_p])\big|_{X_1=0} = 0 \Rightarrow \frac{\partial \boldsymbol{\delta}_1}{\partial X_1}([q_1, \mathbb{Q}_p])\big|_{X_1=0} = 0.$$

Write $q_1 = p^a u$ for $a = \mathrm{ord}_p(q_1)$ and $u \in \mathbb{Z}_p^\times$. Then $\log_p(u) = \log_p(q_1)$. We have

$$\boldsymbol{\delta}_1([q_1, \mathbb{Q}_p]) = \boldsymbol{\delta}_1([p, \mathbb{Q}_p])^a \boldsymbol{\delta}_1([u, \mathbb{Q}_p]) = \boldsymbol{\delta}_1([p, \mathbb{Q}_p])^a (1 + X_1)^{-\log_p(u)/\log_p(\gamma)}$$

(because $\mathcal{N}([u, \mathbb{Q}_p]) = u^{-1}$). Differentiating this identity with respect to X_1, we get from $\boldsymbol{\delta}_1([u, \mathbb{Q}_p])|_{X_1=0} = \boldsymbol{\delta}_1([p, \mathbb{Q}_p])|_{X_1=0} = 1$

$$a\frac{\partial \boldsymbol{\delta}_1([p_1, \mathbb{Q}_p])}{\partial X_1}\Big|_{X_1=0} - \frac{\log_p(q_1)}{\log_p(\gamma)} = a\frac{\partial \boldsymbol{\delta}_1([p_1, \mathbb{Q}_p])}{\partial X_1}\Big|_{X_1=0} - \frac{\log_p(u)}{\log_p(\gamma)} = 0.$$

From this, we conclude

$$\log_p(\gamma)\frac{\partial \boldsymbol{\delta}_1([p_1, \mathbb{Q}_p])}{\partial X_1}\Big|_{X_1=0} = \frac{\log_p(q_1)}{\mathrm{ord}_p(q_1)}.$$

□

The fixed field of the kernel of ϕ_0 is a $\mathbb{Z}_p$-extension $M_\infty/\mathbb{Q}_p$ ($F_1 = \mathbb{Q}_p$). Since $L_1 \ni \phi \mapsto \frac{\phi([\gamma, \mathbb{Q}_p])}{\log_p \gamma} \in \mathbb{Q}_p$ is surjective, M_∞ ramifies fully. Then by local class field theory, $\bigcap_{n=1}^\infty N_{M_n/\mathbb{Q}_p}(M_n^\times)$ has a rank 1 torsion-free part, which contains $q_0 = p^b v$ with $a \neq 0$ and $v \in \mathbb{Z}_p^\times$. The quantity $\frac{\log_p(q_0)}{\mathrm{ord}_p(q_0)} \in \mathbb{Q}_p$ is determined uniquely independent of the choice of q_0, and we now prove

Proposition 1.7.

$$\log_p(\gamma)\frac{\partial \boldsymbol{\delta}_1([p_1, \mathbb{Q}_p])}{\partial X_1}\Big|_{X_1=0} = \frac{\log_p(q_0)}{\mathrm{ord}_p(q_0)}.$$

Proof. Let $\phi_0 = \boldsymbol{\delta}_1^{-1}\frac{\partial \boldsymbol{\delta}_1}{\partial X_1} : D_1^{ab} \to \mathbb{Q}_p$. Let $\mathbf{M}_\infty/\mathbb{Q}_p$ be the composite of all $\mathbb{Z}_p$-extensions of $\mathbb{Q}_p$; so, by local class field theory, $\mathrm{Gal}(\mathbf{M}_\infty/\mathbb{Q}_p) \cong \mathbb{Z}_p^2$. Then $[q_0, \mathbb{Q}_p] \in \mathrm{Gal}(\mathbf{M}_\infty/M_\infty)$ again by local class field theory, and by definition, $\phi_0([q_0, \mathbb{Q}_p]) = 0$. Since $[q_0, \mathbb{Q}_p] = [v, \mathbb{Q}_p][p, \mathbb{Q}_p]^b$ ($b = \mathrm{ord}_p(q_0)$) we have $0 = \phi_0([q_0, \mathbb{Q}_p]) = \phi_0([v, \mathbb{Q}_p]) + b\phi_0([p, \mathbb{Q}_p])$. Writing $M_\infty^{ur}/\mathbb{Q}_p$ for the unique unramified $\mathbb{Z}_p$-extension and $M_\infty^+/\mathbb{Q}_p$ for the cyclotomic $\mathbb{Z}_p$-extension, the restriction of ϕ_0 to $\Gamma^+ = \mathrm{Gal}(M_\infty^+/\mathbb{Q}_p)$ is a constant multiple of $\log_p \circ \mathcal{N}_p$ for the cyclotomic character $\mathcal{N}_p$; i.e., $\phi_0|_{\Gamma^+} = x(\log_p \circ \mathcal{N}_p)$ for $x \in \mathbb{Q}_p^\times$. Since $\log_p(\mathcal{N}_p([v, \mathbb{Q}_p])) = \log_p(v^{-1}) = -\log_p(q_0)$, we have $x\log_p(v^{-1}) + b\phi_0([p, \mathbb{Q}_p]) = 0$. Thus $\mathcal{L}(Ad(T_pE)) = \phi_0([p, \mathbb{Q}_p])/x = \frac{\log_p(q_0)}{\mathrm{ord}_p(q_0)}$. □

1.3. **The non-split case.** We give a detailed proof of Theorem 1.1 when p does not split completely in $F/\mathbb{Q}$.

We prepare some general facts. The following is a slight generalization of [GS1] Section 2: Let K and T be a finite extension of $\mathbb{Q}_p$ and V be a two dimensional vector space over T on which $D := \mathrm{Gal}(\overline{K}/K)$ acts. We write $H^i(?)$ for $H^i(D,?)$. By definition, $H^1(V) = \mathrm{Ext}^1_{T[D]}(T,V)$, and hence, there is a one-to-one correspondence:

$$\begin{Bmatrix}\text{nontrivial extensions}\\ \text{of } T \text{ by } V\end{Bmatrix} \leftrightarrow \begin{Bmatrix}\text{1-dimensional subspaces}\\ \text{of } H^1(V)\end{Bmatrix}.$$

From the left-hand side to the right hand side, the map is given by $(V \hookrightarrow X \twoheadrightarrow T) \mapsto \delta_X(1)$ for the connecting map $T = H^0(T) \xrightarrow{\delta_X} H^1(V)$ of the long exact sequence attached to $(V \hookrightarrow X \twoheadrightarrow T)$. Out of a 1-cocycle $c : D \to V$, one can easily construct an extension $(V \hookrightarrow X \twoheadrightarrow T)$ taking $X = V \oplus T$ and letting D acts on X by $g(v,t) = (gv + t \cdot c(g), t)$, and $[c] \mapsto (V \hookrightarrow X \twoheadrightarrow T)$ gives the inverse map.

By Kummer's theory, we have a canonical isomorphism:

$$H^1(T(1)) \cong \left(\varprojlim_n K^\times/(K^\times)^{p^n}\right) \otimes_{\mathbb{Z}_p} T.$$

We write $\gamma_q \in H^1(T(1))$ for the cohomology class associated to $q \otimes 1$ for $q \in K^\times$. The class γ_q is called the Kummer class of q. A canonical cocycle ξ_q giving the class γ_q is given as follows. Define $\xi_n : D \to \mu_{p^n}$ by $\xi_n(\sigma) = (q^{1/p^n})^{\sigma-1}$, which is a 1-cocycle. Then $\xi_q = \varprojlim_n \xi_n$ having values in $\mathbb{Z}_p(1) \subset T(1)$.

Suppose we have a non-splitting exact sequence of D-modules $0 \to T(1) \to V \to T \to 0$ with the splitting field $\bigcup_n K[\mu_{p^n}, q^{1/p^n}]$ for $q \in K$ with $0 < |q|_p < 1$. We have proven

Proposition 1.8. *If V is isomorphic to the representation $\sigma \mapsto \left(\begin{smallmatrix}\mathcal{N}(\sigma) & \xi_q(\sigma)\\ 0 & 1\end{smallmatrix}\right)$, then for the extension class of $[V] \in H^1(T(1))$, we have $T[V] = T\gamma_q$. In particular, $T\gamma_q$ is in the image of the connecting homomorphism $H^0(T) \xrightarrow{\delta_0} H^1(T(1))$ coming from the extension $T(1) \hookrightarrow V \twoheadrightarrow T$.*

Corollary 1.9. *Let $E_{/K}$ be an elliptic curve. If E has split multiplicative reduction over W, the extension class of $[V]$ for the p-adic Tate module V is in $\mathbb{Q}_p\gamma_{q_E}$ for the Tate period $q_E \in K^\times$.*

Write $\mathcal{D} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \supset D$. We consider $\mathcal{V} = \mathrm{Ind}_K^{\mathbb{Q}_p} V := \mathrm{Ind}_D^{\mathcal{D}} V$. Then we have a D-stable exact sequence $0 \to \mathcal{F}^+V \to V \to V/\mathcal{F}^+V \to 0$ such that D acts by $\mathcal{N}$ on $\mathcal{F}^+V$. Thus $\mathcal{F}^+V$ is one dimensional. We then have the exact sequence of the induced modules:

$$0 \to \mathrm{Ind}_K^{\mathbb{Q}_p} \mathcal{F}^+V \to \mathrm{Ind}_K^{\mathbb{Q}_p} V \to \mathrm{Ind}_K^{\mathbb{Q}_p}(V/\mathcal{F}^+V) \to 0.$$

We put $\mathcal{F}^+\mathcal{V} := \mathrm{Ind}_K^{\mathbb{Q}_p} \mathcal{F}^+V$, and define $\mathcal{F}^{00}\mathcal{V}$ by the maximal subspace of $\mathcal{V}$ stable under $\mathcal{D}$ such that $\mathcal{D}$ acts on $\mathcal{F}^{00}\mathcal{V}/\mathcal{F}^+\mathcal{V}$ trivially. In other words, we have

$$H^0(\mathcal{D}, \mathcal{V}/\mathcal{F}^+\mathcal{V}) = \mathcal{F}^{00}\mathcal{V}/\mathcal{F}^+\mathcal{V}.$$

Similarly, we define $\mathcal{F}^{11}\mathcal{V} \subset \mathcal{V}$ to be the smallest subspace stable under $\mathcal{D}$ such that $\mathcal{D}$ acts on $\mathcal{F}^+\mathcal{V}/\mathcal{F}^{11}\mathcal{V}$ by $\mathcal{N}$; so, we have

$$H_0(\mathcal{D}, \mathcal{F}^+\mathcal{V}(-1)) = (\mathcal{F}^+\mathcal{V}/\mathcal{F}^{11}\mathcal{V})(-1).$$

Since $\mathrm{Ind}_K^{\mathbb{Q}_p}(V/\mathcal{F}^+V) \cong \mathrm{Ind}_K^{\mathbb{Q}_p} \mathbf{1}$ and $\mathrm{Ind}_K^{\mathbb{Q}_p} \mathcal{F}^+V \cong \mathrm{Ind}_K^{\mathbb{Q}_p} \mathcal{F}^+\mathcal{N} \cong (\mathrm{Ind}_K^{\mathbb{Q}_p} \mathbf{1}) \otimes \mathcal{N}$, we find $\dim_T(\mathcal{F}^+\mathcal{V}/\mathcal{F}^{11}\mathcal{V}) = \dim_T(\mathcal{F}^{00}\mathcal{V}/\mathcal{F}^+\mathcal{V}) = 1$, because $H^0(\mathcal{D}, \mathrm{Ind}_K^{\mathbb{Q}_p} \mathbf{1}) \cong H_0(\mathcal{D}, \mathrm{Ind}_K^{\mathbb{Q}_p} \mathbf{1}) \cong T$. Thus we get an extension

$$0 \to \mathcal{F}^+\mathcal{V}/\mathcal{F}^{11}\mathcal{V} \to \mathcal{F}^{00}\mathcal{V}/\mathcal{F}^{11}\mathcal{V} \to \mathcal{F}^{00}\mathcal{V}/\mathcal{F}^+\mathcal{V} \to 0 \tag{1.2}$$

of $T[\mathcal{D}]$-modules.

Let $\widetilde{T} := T[\varepsilon] = T[t]/(t^2)$ with $\varepsilon \leftrightarrow (t \mod t^2)$. A $\widetilde{T}[D]$-module $\widetilde{V}$ is called an infinitesimal deformation of V if $\widetilde{V}$ is $\widetilde{T}$-free of rank 2 and $\widetilde{V}/\varepsilon\widetilde{V} \cong V$ as $T[D]$-modules. Since the map $\varepsilon : \widetilde{V} \twoheadrightarrow V \subset \widetilde{V}$ given by $v \mapsto \varepsilon v$ is Galois equivariant, we have an exact sequence of D-modules

$$0 \to V \to \widetilde{V} \to V \to 0$$

if $V[\varphi]$ is an infinitesimal deformation of V. Pick an infinitesimal character $\psi : D \to \widetilde{T}^\times$ with $\psi \mod (\varepsilon) = 1$. Define $\widetilde{T}(\psi)$ for the space of the character ψ. Obviously, $\frac{d\psi}{d\varepsilon} : D \to T$ is a homomorphism; so, $\frac{d\psi}{d\varepsilon} \in \mathrm{Hom}(D, T) = H^1(T)$. Since the extension $\widetilde{V}$ is split if and only if $\frac{d\psi}{d\varepsilon} = 0$, we get

Proposition 1.10. *The correspondence $\widetilde{T}(\psi) \leftrightarrow \frac{d\psi}{d\varepsilon} \in H^1(T)$ gives a one-to-one correspondence:*

$$\left\{\begin{matrix}\textit{Nontrivial infinitesimal}\\ \textit{deformations of } T\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\textit{1-dimensional}\\ \textit{subspaces of } H^1(T)\end{matrix}\right\},$$

and we have $T[\widetilde{V}(\psi)] = T\frac{d\psi}{d\varepsilon}$ in $H^1(T)$.

We have the restriction map $\mathrm{Res} : H^1(\mathcal{D}, T(m)) \to H^1(T(m))$ and the transfer map $\mathrm{Tr} : H^1(T(m)) \to H^1(\mathcal{D}, T(m))$. We have the cup product pairing giving Tate duality and the following commutative diagram:

$$\begin{array}{cccccccc} \langle\cdot,\cdot\rangle : & H^1(T(1)) & \times & H^1(T) & \to & H^2(T(1)) & \cong T \\ & \mathrm{Tr} \downarrow & & \uparrow \mathrm{Res} & & \| & \\ \langle\cdot,\cdot\rangle : & H^1(\mathcal{D}, T(1)) & \times & H^1(\mathcal{D}, T) & \to & H^2(\mathcal{D}, T(1)) & \cong T. \end{array}$$

By Shapiro's lemma (and the Frobenius reciprocity; cf., [HMI] Section 3.4.4), we get

Lemma 1.11. *We have $\mathrm{Tr}([V]) = [\mathcal{F}^{00}\mathcal{V}/\mathcal{F}^{11}\mathcal{V}] \in H^1(\mathcal{D}, T(1))$ for the class $[V] \in H^1(T(1))$ of the extension $T(1) \hookrightarrow V \twoheadrightarrow T$.*

Proof. Decompose $\mathcal{D} = \bigsqcup_{\sigma\in\Sigma} D\sigma$; so, $\Sigma \cong \mathrm{Hom}_{\text{field}}(K, \overline{\mathbb{Q}}_p)$. Then for $\tau \in \mathcal{D}$, we have $\sigma\tau = \tau_\sigma\sigma'$ for $\sigma' \in \Sigma$ and $\tau_\sigma \in D$. We look at the matrix form of the induced representation. If the matrix form of V is given by $\left(\begin{smallmatrix}\mathcal{N} & \xi \\ 0 & 1\end{smallmatrix}\right)$ for a 1-cocycle $\xi : D \to T(1)$, the cocycle giving the extension $T(1) \hookrightarrow \mathcal{F}^{00}\mathcal{V}/\mathcal{F}^{11}\mathcal{V} \twoheadrightarrow T(1)$ is given by $\tau \mapsto \sum_{\sigma\in\Sigma} \xi(\tau_\sigma)^\sigma$, which represents the class of $\mathrm{Tr}([\xi])$. Here $\mathcal{D}$ acts on the right on $\mathbb{Z}_p(1) = \varprojlim_n \mu_{p^n}$ following the tradition of Galois action on roots of unity $\zeta \mapsto \zeta^\sigma$. □

Corollary 1.12. *Let $E_{/K}$ be an elliptic curve. If E has split multiplicative reduction over W, the extension class of $[\mathcal{F}^{00}\mathcal{V}/\mathcal{F}^{11}\mathcal{V}]$ for $\mathcal{V} = \mathrm{Ind}_K^{\mathbb{Q}_p} V$ with the p-adic Tate module V is in $\mathbb{Q}_p\gamma_{N_{K/\mathbb{Q}_p}(q_E)}$ for the Tate period $q_E \in K^\times$.*

Proof. We keep the notation introduced in the proof of the above lemma. Consider the cocycle $\xi_n(\tau) = (q_E^{1/p^n})^{\tau-1}$ of D with values in μ_{p^n}. Then we have

$$\mathrm{Tr}(\xi_n)(\sigma) = \prod_{\sigma\in\Sigma} (q_E^{1/p^n})^{(\tau_\sigma-1)\sigma} = \prod_{\sigma\in\Sigma} (q_E^{1/p^n})^{\sigma(\tau-1)} = ((N_{K/\mathbb{Q}_p} q_E)^{1/p^n})^{\tau-1}.$$

Thus $\mathrm{Tr}([V]) = [\mathcal{F}^{00}\mathcal{V}/\mathcal{F}^{11}\mathcal{V}]$ is represented by the cocycle ξ given by $\lim_n \mathrm{Tr}(\xi_n)$ for $\mathrm{Tr}(\xi_n)(\tau) = (N_{K/\mathbb{Q}_p}(q_E)^{1/p^n})^{\tau-1}$, which implies that $\mathrm{Tr}([V]) = \gamma_{N_{K/\mathbb{Q}_p}(q_E)}$. □

Note that

$$H^1(\mathcal{D}, T) \cong \mathrm{Hom}(\mathcal{D}, T) = \mathrm{Hom}(\mathcal{D}^{ab}, T) \cong T^2,$$

where the last isomorphism is given by

$$\mathrm{Hom}(\mathcal{D}^{ab}, T) \ni \phi \mapsto \left(\frac{\phi([\gamma, \mathbb{Q}_p])}{\log_p(\gamma)}, \phi([p, \mathbb{Q}_p])\right) \in T^2$$

for $\gamma \in \mathbb{Z}_p^\times$ of infinite order. This follows from class field theory and $[x, \mathbb{Q}_p]$ for $x \in \mathbb{Q}_p^\times$ is the local Artin symbol. Since the duality is perfect, for any line L in $H^1(\mathcal{D}, T)$, one can assign its orthogonal complement $L^\perp$ in $H^1(\mathcal{D}, T(1))$ under the Tate duality $\langle\cdot,\cdot\rangle$. Thus we have

Proposition 1.13. *Suppose $K = \mathbb{Q}_p$. The correspondence of a line in $H^1(\mathcal{D}, T)$ and its orthogonal complement in $H^1(\mathcal{D}, T(1))$ gives a one-to-one correspondence:*

$$\left\{\begin{matrix}\textit{Nontrivial extensions} \\ \textit{of } T \textit{ by } T(1) \textit{ as } T[\mathcal{D}]\textit{-modules}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\textit{nontrivial infinitesimal} \\ \textit{deformations of } T \textit{ over } \mathcal{D}\end{matrix}\right\}.$$

Let $\sigma_q = [q, \mathbb{Q}_p]^{-1}$ for the Artin symbol $[x, \mathbb{Q}_p]$ normalized so that $\mathcal{N}([u, \mathbb{Q}_p]) = u^{-1}$ for $u \in \mathbb{Z}_p^\times$ and $[p, \mathbb{Q}_p]$ is the arithmetic Frobenius element. Then we have $\langle\gamma_q, \xi\rangle = \xi(\sigma_q)$ for $\gamma_q \in H^1(\mathcal{D}, \mathbb{Q}_p(1))$ and $\xi \in \mathrm{Hom}(\mathcal{D}, \mathbb{Q}_p) = H^1(\mathcal{D}, \mathbb{Q}_p)$. Now we are ready to prove the following version of a theorem of Greenberg-Stevens (cf. [GS1] 2.3.4):

Theorem 1.14. *Let $E_{/K}$ be an elliptic curve with split multiplicative reduction and let $\psi : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \widetilde{\mathbb{Q}_p}^\times$ be a nontrivial character which is $\equiv 1$ modulo ε. Let V*

be the p-adic Tate module of E, $\mathcal{V}$ be the induced Galois representation $\mathrm{Ind}_K^{\mathbb{Q}_p} V$ *and* $q_E \in K^\times$ *be the Tate period of E. Then the following statements are equivalent:*

(a) $\frac{d\psi}{d\varepsilon}(\sigma_{N_{K/\mathbb{Q}_p}(q_E)}) = 0$;

(b) $\mathcal{W} := \mathcal{F}^{00}\mathcal{V}/\mathcal{F}^{11}\mathcal{V}$ *corresponds to* $\widetilde{\mathbb{Q}_p}(\psi)$ *under the correspondence of* Proposition 1.13*;*

(c) *There is an infinitesimal deformation* $\widetilde{\mathcal{W}}$ *of* $\mathcal{W}$ *and a commutative diagram:*

$$\begin{array}{ccccc} \widetilde{\mathbb{Q}_p}(1) & \overset{\hookrightarrow}{\longrightarrow} & \widetilde{\mathcal{W}} & \overset{\twoheadrightarrow}{\longrightarrow} & \widetilde{\mathbb{Q}_p}(\psi) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{Q}_p(1) & \underset{\hookrightarrow}{\longrightarrow} & \mathcal{W} & \underset{\twoheadrightarrow}{\longrightarrow} & \mathbb{Q}_p, \end{array}$$

in which the top row is an exact sequence of $\widetilde{\mathbb{Q}_p}[\mathcal{D}]$*-modules and the vertical map is the reduction modulo* ε*.*

Proof. Since $\langle \gamma_q, \xi \rangle = \xi(\sigma_q)$ for $\xi \in H^1(\mathcal{D}, \mathbb{Q}_p) = \mathrm{Hom}(\mathcal{D}, \mathbb{Q}_p)$ and $\gamma_q \in H^1(\mathcal{D}, \mathbb{Q}_p(1))$, applying these formulas to $\xi = \frac{d\psi}{d\varepsilon}$, we get (a) $\Leftrightarrow$ (b) by the definition of the correspondence in Proposition 1.13.

The equivalence (b) $\Leftrightarrow$ (c) can be proven in exactly the same manner as in the proof of [GS1] 2.3.4. Here is the argument proving (b) $\Rightarrow$ (c). Let c be a 1-cocycle representing γ_Q for $Q = N_{K/\mathbb{Q}_p}(q_E)$. Then $\mathcal{D} \times \mathcal{D} \ni (\sigma, \tau) \mapsto c(\sigma)\frac{d\psi}{d\varepsilon}(\tau) \in \mathbb{Q}_p(1)$ is the 2-cocycle representing the cup product $\gamma_Q \cup [\widetilde{\mathbb{Q}_p}(\psi)]$, which vanishes by (b). Thus it is a 2-coboundary:

$$c(\sigma)\frac{d\psi}{d\varepsilon}(\tau) = \partial\xi(\sigma, \tau) = \xi(\sigma\tau) - \mathcal{N}(\sigma)\xi(\tau) - \xi(\sigma)$$

for a 1-chain $\xi : \mathcal{D} \to \mathbb{Q}_p(1)$. Then defining an action of $\sigma \in \mathcal{D}$ on $\widetilde{\mathbb{Q}_p}^2$ via the matrix multiplication by $\begin{pmatrix} \mathcal{N}(\sigma) & c(\sigma)+\xi(\sigma)\varepsilon \\ 0 & \psi(\sigma) \end{pmatrix}$, the resulting $\widetilde{\mathbb{Q}_p}[\mathcal{D}]$-module $\widetilde{\mathcal{W}}$ fits well in the diagram in (c).

Conversely suppose we have the commutative diagram as in (c), which can be written as the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Q}_p(1) & \longrightarrow & \mathcal{W} & \longrightarrow & \mathbb{Q}_p & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \widetilde{\mathbb{Q}_p}(1) & \longrightarrow & \widetilde{\mathcal{W}} & \longrightarrow & \widetilde{\mathbb{Q}_p} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Q}_p(1) & \longrightarrow & \mathcal{W} & \longrightarrow & \mathbb{Q}_p & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

The connecting homomorphism $d : H^1(\mathcal{D}, \mathbb{Q}_p(1)) \to H^2(\mathcal{D}, \mathbb{Q}_p(1))$ vanishes because the leftmost vertical sequence splits. On the other hand, letting $\delta_\psi : H^0(\mathcal{D}, \mathbb{Q}_p) \to H^1(\mathcal{D}, \mathbb{Q}_p)$ stand for the connecting homomorphism of degree 0 coming from the rightmost vertical sequence, and letting $\delta_i : H^i(\mathcal{D}, \mathbb{Q}_p) \to H^{i+1}(\mathcal{D}, \mathbb{Q}_p(1))$ be the connecting homomorphism of degree i associated to the bottom row (and also to the top row). By the commutativity of the diagram, we get the following commutative square:

$$\begin{array}{ccc} H^0(\mathcal{D}, \mathbb{Q}_p) = \mathbb{Q}_p & \xrightarrow{\delta_0} & H^1(\mathcal{D}, \mathbb{Q}_p(1)) \\ {\scriptstyle \delta_\psi}\downarrow & & \downarrow{\scriptstyle d=0} \\ H^1(\mathcal{D}, \mathbb{Q}_p) & \xrightarrow[\delta_1]{} & H^2(\mathcal{D}, \mathbb{Q}_p(1)). \end{array}$$

Since $\delta_\psi(1) = \frac{d\psi}{d\varepsilon}$, we confirm $\frac{d\psi}{d\varepsilon} \in \mathrm{Ker}(\delta_1)$. By Proposition 1.8, γ_Q is the in the image of δ_0. Thus the assertion (b) follows if we can show that $\mathrm{Ker}(\delta_1)$ is orthogonal to $\mathrm{Im}(\delta_0)$.

Since $\mathcal{V} = \mathrm{Ind}_K^{\mathbb{Q}} V$ is the p-adic Tate module of the principally polarized abelian variety $A = \mathrm{Res}_{K/\mathbb{Q}_p} E_{/K}$ (the Weil restriction), $\mathcal{V}$ has self dual under the polarization pairing, which induces a self duality of $\mathcal{W}$ and also the self (Cartier) duality of the exact sequence $0 \to \mathbb{Q}_p(1) \to \mathcal{W} \to \mathbb{Q}_p \to 1$. In particular the inclusion $\iota : \mathbb{Q}_p(1) \to \mathcal{W}$ and the projection $\pi : \mathcal{W} \twoheadrightarrow \mathbb{Q}_p(1)$ are mutually adjoint under the pairing. Thus the connecting maps $\delta_0 : H^0(\mathcal{D}, \mathbb{Q}_p) \to H^1(\mathcal{D}, \mathbb{Q}_p(1))$ and $\delta_1 : H^1(\mathcal{D}, \mathbb{Q}_p) \to H^2(\mathcal{D}, \mathbb{Q}_p(1))$ are mutually adjoint each other under the Tate duality pairing. In particular, $\mathrm{Im}(\delta_0)$ is orthogonal to $\mathrm{Ker}(\delta_1)$. □

Take a prime $\mathfrak{p}|p$ in F, and let $D = \mathrm{Gal}(\overline{F}_\mathfrak{p}/F_\mathfrak{p})$ and $\mathcal{D} = \mathrm{Gal}(\overline{F}_\mathfrak{p}/\mathbb{Q}_p)$. We write $\mathcal{I}$ (resp. I) for the inertia group of $\mathcal{D}$ (resp. D).

Lemma 1.15. *Let $\rho_A : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to GL_2(A)$ be a deformation of ρ_F for an artinian local K-algebra A with residue field K. Write $\rho_A|_D = \left(\begin{smallmatrix} \varepsilon_A & * \\ 0 & \delta_A \end{smallmatrix}\right)$ with $\delta_A \equiv \alpha_\mathfrak{p} \mod \mathfrak{m}_A$. Suppose that $\alpha_\mathfrak{p}$ can be extended to a character $\widetilde{\alpha}_\mathfrak{p} : \mathcal{D} \to K^\times$. If $\delta_A|_I$ factors through $\mathrm{Gal}(F_\mathfrak{p}[\mu_{p^\infty}]/F_\mathfrak{p})$, the character δ_A extends to a unique character $\widetilde{\delta}_A$ of $\mathcal{D}$ with values in $A^\times$ such that $\widetilde{\delta}_A \equiv \widetilde{\alpha}_\mathfrak{p} \mod \mathfrak{m}_A$.*

Proof. Let $F_\mathfrak{p}^{ab}$ (resp. $F_\mathfrak{p}^{ur}$) be the maximal abelian extension of $F_\mathfrak{p}$ (resp. the maximal unramified extension of $F_\mathfrak{p}$). Then we have

$$F_\mathfrak{p}[\mu_{p^\infty}] \subset F_\mathfrak{p}^{ur}[\mu_{p^\infty}] = F_\mathfrak{p}\mathbb{Q}_p^{ur}[\mu_{p^\infty}] = F_\mathfrak{p}\mathbb{Q}_p^{ab}.$$

Thus $\mathrm{Gal}(F_\mathfrak{p}\mathbb{Q}_p^{ur}[\mu_{p^\infty}]/F_\mathfrak{p})$ can be identified with the subgroup $\mathrm{Gal}(\mathbb{Q}_p^{ab}/\mathbb{Q}_p^{ab} \cap F_\mathfrak{p})$ of $\mathrm{Gal}(\mathbb{Q}_p^{ab}/\mathbb{Q}_p)$ of finite index. Since δ_A is a character of $\mathrm{Gal}(F_\mathfrak{p}\mathbb{Q}_p^{ur}[\mu_{p^\infty}]/F_\mathfrak{p})$, regarding it as a character of $\mathrm{Gal}(\mathbb{Q}_p^{ab}/\mathbb{Q}_p^{ab} \cap F_\mathfrak{p})$, we only need to extend it to $\mathrm{Gal}(\mathbb{Q}_p^{ab}/\mathbb{Q}_p)$. Since $F_\mathfrak{p} \cap \mathbb{Q}_p^{ab}/\mathbb{Q}_p$ is a finite Galois extension with an abelian Galois group Δ, by the theory of the Schur multiplier, the obstruction of extending character lies in

$H^2(\Delta, A^\times)$ (see [MFG] Section 3.3.5). Since $\delta_A \equiv \alpha_{\mathfrak{p}} \mod \mathfrak{m}_A$, the obstruction class $Ob(\delta_A) \equiv Ob(\alpha_{\mathfrak{p}}) = 0 \mod \mathfrak{m}_A$. Thus $Ob(\delta_A) \in H^2(\Delta, 1+\mathfrak{m}_A)$. Since $1+\mathfrak{m}_A$ is uniquely divisible (by $\log : 1+\mathfrak{m}_A \cong \mathfrak{m}_A$ as K-vector space), we get the vanishing $H^2(\Delta, 1+\mathfrak{m}_A) = 0$ for the finite group Δ. Then we can extend δ_A to $\widetilde{\delta}_A$ with $\widetilde{\delta}_A \equiv \widetilde{\alpha}_{\mathfrak{p}}$ mod $\mathfrak{m}_A$ as proven in [MFG] Section 5.4. If δ' is another extension, we find $\widetilde{\delta}_A^{-1}\delta'$ is a character of Δ, which has to be trivial by the condition $\widetilde{\delta}_A \cong \widetilde{\alpha}_{\mathfrak{p}} \mod \mathfrak{m}_A$. Thus the extension is unique. □

We recall the theorem in the general case. Order the prime factors of p as $\mathfrak{p}_1, \ldots, \mathfrak{p}_g$. We write $F_i = F_{\mathfrak{p}_i}$ and N_i for the norm map $N_{F_i/\mathbb{Q}_p} : F_i \to \mathbb{Q}_p$. Here we do not assume that p splits completely in $F/\mathbb{Q}$. Take an elliptic curve $E_{/F}$. If E is split multiplicative at $\mathfrak{p}_j$ for $j = 1, 2, \ldots, k$ ($k \le g$) with Tate period $q_i \in F_i$ at $\mathfrak{p}_i$ for $i \le k$ and having ordinary good reduction at $\mathfrak{p}_i$ with $i > k$, we find

Theorem 1.16. *Assume that $R \cong \mathbb{Q}_p[[X_{\mathfrak{p}}]]_{\mathfrak{p}|p}$. Then for the local Artin symbol $[p, F_{\mathfrak{p}}]$, we have for $\rho_E = T_pE$*

$$\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_E)) = \left(\prod_{i=1}^{k} \frac{\log_p(N_i(q_i))}{\mathrm{ord}_p(N_i(q_i))}\right) \det\left(\frac{\partial \boldsymbol{\delta}_{\mathfrak{p}}([p, F_i])}{\partial X_j}\right)_{i>k, j>k} \Big|_{X=0} \prod_{i>k} \frac{\log_p(\gamma_{\mathfrak{p}_i})}{\alpha_{\mathfrak{p}}([p, F_i])},$$

where $\gamma_{\mathfrak{p}}$ is the generator of $\mathcal{N}(\mathrm{Gal}(F_{\mathfrak{p}}[\mu_{p^\infty}]/F_{\mathfrak{p}}))$ by which we identify the group algebra $W[[\Gamma_{\mathfrak{p}}]]$ with $W[[X_{\mathfrak{p}}]]$.

Proof. By the same argument which proves the formula (1.1) (taking the locally cyclotomic Hecke algebra introduced in [HMI] Section 3.2.9), we get

$$\begin{aligned}(1.3)\quad \mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_E)) = &\prod_{i=1}^{k} \frac{\partial \boldsymbol{\delta}_{\mathfrak{p}}([p, F_i])}{\partial X_i}\Big|_{X_i=0} \log_p(\gamma_{\mathfrak{p}_i})\alpha_{\mathfrak{p}_i}([p, F_i])^{-1} \\ &\times \det\left(\frac{\partial \boldsymbol{\delta}_{\mathfrak{p}}([p, F_i])}{\partial X_j}\right)_{i\ge k, j\ge k} \Big|_{X=0} \prod_{j\ge k} \log_p(\gamma_{\mathfrak{p}_j})\alpha_{\mathfrak{p}}([p, F_j])^{-1}.\end{aligned}$$

Let $V = T_pE \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ for the p-adic Tate module T_pE of E. The global representation $\mathbb{V} = \mathrm{Ind}_F^{\mathbb{Q}} V$ has decreasing filtration $\mathcal{F}^i\mathbb{V}$ such that an open subgroup of the inertia group I_p at p acts on $\mathcal{F}^i\mathbb{V}/\mathcal{F}^{i+1}\mathbb{V}$ by the i-th power of the cyclotomic character $\mathcal{N}$ and $\mathcal{F}^1\mathbb{V} \subsetneq \mathbb{V}$. Put $\mathcal{F}^+\mathbb{V} = \mathcal{F}^1\mathbb{V}$. Write $\mathcal{D} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. Let $\mathcal{F}^{00}\mathbb{V}$ be the maximal $\mathcal{D}$-stable subspace of $\mathbb{V}$ containing $\mathcal{F}^+\mathbb{V}$ such that any vector in $\mathcal{F}^{00}\mathbb{V}/\mathcal{F}^+\mathbb{V}$ is fixed by $\mathcal{D}$. Similarly, let $\mathcal{F}^{11}\mathbb{V}$ be the minimal $\mathcal{D}$-stable subspace of $\mathbb{V}$ contained in $\mathcal{F}^+\mathbb{V}$ such that $\mathcal{D}$ acts on $\mathcal{F}^+\mathbb{V}/\mathcal{F}^{11}\mathbb{V}$ by $\mathcal{N}$. We may regard V as a $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F_j)$-module, and consider $\mathcal{V}_j = \mathrm{Ind}_{F_j}^{\mathbb{Q}_p} V$. Then again we have $\mathcal{F}^{00}\mathcal{V}_j \supset \mathcal{F}^{11}\mathcal{V}_j$ as defined above

(1.2) for $K = F_j$. From [HMI] (3.4.4), we see easily that

$$\mathcal{F}^{00}\mathbb{V}/\mathcal{F}^{11}\mathbb{V} \cong \bigoplus_{j=1}^{k} \frac{\mathcal{F}^{00}\mathcal{V}_j}{\mathcal{F}^{11}\mathcal{V}_j}$$

as $\mathcal{D}$-modules. Because of this decomposition, we can fix j and only need to compute the $\mathcal{L}$-invariant for the j-th factor. We write $D = \mathrm{Gal}(\overline{\mathbb{Q}}_p/F_j) \subset \mathcal{D}$. We consider the universal locally cyclotomic deformation $\boldsymbol{\rho}$ of V under the conditions (K1–4) and consider $\widetilde{V}_j = (\boldsymbol{\rho} \mod \mathfrak{m}_j)$ for $\mathfrak{m}_j := (X_1, \dots, X_{j-1}, X_j^2, X_{j+1}, \dots, X_g) \subset \mathbb{Q}_p[[X_j]]_{j=1,\dots,g}$. Again we consider $\widetilde{\mathcal{V}}_j := \mathrm{Ind}_{F_j}^{\mathbb{Q}_p} \widetilde{V}_j$. We put $\mathcal{F}^+\widetilde{\mathcal{V}}_j = \mathrm{Ind}_{F_j}^{\mathbb{Q}_p} \mathcal{F}^+\widetilde{V}_j$. We have D-stable filtration $\mathcal{F}^+\widetilde{V}_j \subset \widetilde{V}_j$ such that D acts on $\mathcal{F}^+\widetilde{V}_j\backslash\widetilde{V}_j$ by the nearly ordinary character

$$\delta_j := (\boldsymbol{\delta}_j \mod (X_1, \dots, X_{j-1}, X_j^2, X_{j+1}, \dots, X_g)).$$

The character δ_j satisfies $\delta_j \equiv \alpha_{\mathfrak{p}_j} = \mathbf{1} \mod (X_j)$ for the trivial character $\mathbf{1}$ of D. Since $\alpha_{\mathfrak{p}_j}$ can be extended to $\mathbf{1} : \mathcal{D} \to \mathbb{Q}_p^\times$, by Lemma 1.15, δ_j has a unique extension $\widetilde{\delta}_j : \mathcal{D} \to \widetilde{\mathbb{Q}_p}^\times$ with $\widetilde{\delta}_j \equiv \mathbf{1} \mod (X_j)$ (identifying $\widetilde{\mathbb{Q}_p}$ with $\mathbb{Q}_p[X_j]/(X_j)^2$). $\mathrm{Ind}_{F_j}^{\mathbb{Q}_p} \delta_j \cong \widetilde{\delta}_j \otimes \mathrm{Ind}_{F_j}^{\mathbb{Q}_p} \mathbf{1}$. Thus we have a unique subspace $\mathcal{F}^{00}\widetilde{\mathcal{V}}_j \subset \widetilde{\mathcal{V}}_j$ such that $\mathcal{F}^{00}\widetilde{\mathcal{V}}_j/\mathcal{F}^+\widetilde{\mathcal{V}}_j = H^0(\mathcal{D}, \widetilde{\mathcal{V}}_j/\mathcal{F}^+\widetilde{\mathcal{V}}_j(\widetilde{\delta}_j^{-1}))$. The $\widetilde{\mathbb{Q}_p}$-module $\mathcal{F}^{00}\widetilde{\mathcal{V}}_j/\mathrm{Ind}_{F_j}^{\mathbb{Q}_p} \mathcal{F}^+\widetilde{V}_j$ is free of rank 1 over $\widetilde{\mathbb{Q}_p}$.

Write $(\boldsymbol{\rho}|_D)^{ss} = \boldsymbol{\delta}_j \oplus \boldsymbol{\epsilon}_j$, and define again

$$\epsilon_j := (\boldsymbol{\epsilon}_j \mod (X_1, \dots, X_{j-1}, X_j^2, X_{j+1}, \dots, X_g)).$$

Then $\epsilon_j \equiv \mathcal{N} \mod (X_j)$, and again applying Lemma 1.15 to ϵ_j, it has a unique extension $\widetilde{\epsilon}_j : \mathcal{D} \to \widetilde{\mathbb{Q}_p}^\times$ with $\widetilde{\epsilon}_j \equiv \mathcal{N} \mod (X_j)$. Thus $\mathrm{Ind}_{F_j}^{\mathbb{Q}_p} \mathcal{F}^+\widetilde{V}_j = \widetilde{\epsilon}_j \otimes \mathrm{Ind}_{F_j}^{\mathbb{Q}_p} \mathbf{1}$. Then we have a unique subspace $\mathcal{F}^{11}\widetilde{\mathcal{V}}_j \subset \mathcal{F}^+\widetilde{\mathcal{V}}_j$ such that $H_0(\mathcal{D}, \mathcal{F}^+\widetilde{\mathcal{V}}_j(\widetilde{\epsilon}_j^{-1})) = \mathcal{F}^+\widetilde{\mathcal{V}}_j/\mathcal{F}^{11}\widetilde{\mathcal{V}}_j$. Again $\mathcal{F}^+\widetilde{\mathcal{V}}_j/\mathcal{F}^{11}\widetilde{\mathcal{V}}_j$ is $\widetilde{\mathbb{Q}_p}$-free of rank 1. By the uniqueness of the extensions, we have $\widetilde{\delta}_j\widetilde{\epsilon}_j = \mathcal{N}$ over $\mathcal{D}$, because $\delta_j\epsilon_j = \mathcal{N}$ over D.

Since we have the D-equivariant duality pairing $\widetilde{V}_j \times \widetilde{V}_j \to \widetilde{\mathbb{Q}_p}(1)$ by the fixed determinant condition, the duality extends to $\mathcal{D}$-equivariant duality pairing $\widetilde{\mathcal{V}}_j \times \widetilde{\mathcal{V}}_j \to \widetilde{\mathbb{Q}_p}(1)$, and we have $\mathcal{F}^{11}\widetilde{\mathcal{V}}_j \subset \mathrm{Ind}_{F_j}^{\mathbb{Q}_p} \mathcal{F}^+\widetilde{V}_j$ by $(\mathcal{F}^{00}\widetilde{\mathcal{V}}_j)^\perp$. The matrix form of the $\mathcal{D}$-representation: $\mathcal{F}^{00}\widetilde{\mathcal{V}}_j/\mathcal{F}^{11}\widetilde{\mathcal{V}}_j$ is $\left(\begin{smallmatrix} \widetilde{\epsilon}_j & * \\ 0 & \widetilde{\delta}_j \end{smallmatrix}\right)$. Twist $\mathcal{F}^{00}\widetilde{\mathcal{V}}_j/\mathcal{F}^{11}\widetilde{\mathcal{V}}_j$ by $\chi = \widetilde{\epsilon}_j^{-1}\mathcal{N}$; then, $\mathcal{F}^{00}\widetilde{\mathcal{V}}_j/\mathcal{F}^{11}\widetilde{\mathcal{V}}_j(\chi)$ has the matrix form $\left(\begin{smallmatrix} \mathcal{N} & * \\ 0 & \psi_j \end{smallmatrix}\right)$ for $\psi_j = \widetilde{\delta}_j\widetilde{\epsilon}_j^{-1}\mathcal{N}$. Since $\det \boldsymbol{\rho} = \mathcal{N}$, we have $\widetilde{\delta}_j\widetilde{\epsilon}_j = \mathcal{N}$, and hence $\psi_j = \widetilde{\delta}_j^2$. Then $\mathcal{F}^{00}\widetilde{\mathcal{V}}_j/\mathcal{F}^{11}\widetilde{\mathcal{V}}_j$ is an infinitesimal extension of $\mathcal{F}^{00}\mathcal{V}_j/\mathcal{F}^{11}\mathcal{V}_j$ making the following diagram commutative:

$$\begin{array}{ccccc} \widetilde{\mathbb{Q}_p}(1) & \xrightarrow{\hookrightarrow} & \mathcal{F}^{00}\widetilde{\mathcal{V}}_j/\mathcal{F}^{11}\widetilde{\mathcal{V}}_j(\chi) & \xrightarrow{\twoheadrightarrow} & \widetilde{\mathbb{Q}_p}(\psi_j) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{Q}_p(1) & \xrightarrow[\hookrightarrow]{} & \mathcal{F}^{00}\mathcal{V}_j/\mathcal{F}^{11}\mathcal{V}_j & \xrightarrow[\twoheadrightarrow]{} & \mathbb{Q}_p. \end{array}$$

This diagram satisfies the condition (c) of Theorem 1.14, and

$$\frac{\partial\psi_j}{\partial X_j}|_{X_j=0}([N_j(q_j),\mathbb{Q}_p]) = 2\widetilde{\delta}_j\frac{\partial\widetilde{\delta}_j}{\partial X_j}|_{X_j=0}([N_j(q_j),\mathbb{Q}_p]) = 0$$

$$\Rightarrow \frac{\partial\widetilde{\delta}_j}{\partial X_j}|_{X_j=0}([N_j(q_j),\mathbb{Q}_p]) = 0.$$

Write $N_j(q_j) = p^a u$ for $a = \mathrm{ord}_p(N_j(q_j))$ and $u \in \mathbb{Z}_p^\times$. Then $\log_p(u) = \log_p(N_j(q_j))$. Write $d_j = [F_j : \mathbb{Q}_p]$. Since $[p,\mathbb{Q}_p]^{d_j} = [N_j(p),\mathbb{Q}_p] = [p,F_j]|_{\mathbb{Q}_p^{ab}}$ and $[u,\mathbb{Q}_p]^{d_j} = [N_j(u),\mathbb{Q}_p] = [u,F_j]|_{\mathbb{Q}_p^{ab}}$, we have

$$\widetilde{\delta}_j([N(q_j),\mathbb{Q}_p]^{d_j}) = \delta_j([p,F_j])^a\delta_j([u,F_j]) = \delta_j([p,F_j])^a(1+X_j)^{-\log_p(u)/\log_p(\gamma_{\mathfrak{p}_j})}$$

(because $\mathcal{N}([u,\mathbb{Q}_p]) = u^{-1}$). Differentiating this identity with respect to X_j, we get from $\delta_j([u,F_j])|_{X_j=0} = \delta_j([p,F_j])|_{X_j=0} = \alpha_{\mathfrak{p}_j}([p,F_j]) = 1$

$$a\frac{\partial\boldsymbol{\delta}_j}{\partial X_j}\Big|_{X_j=0}([p,F_j]) - \frac{\log_p(u)}{\log_p(\gamma_{\mathfrak{p}_j})} = 0$$

From this we conclude

$$\frac{\partial\boldsymbol{\delta}_{\mathfrak{p}}([p,F_j])}{\partial X_j}\Big|_{X_j=0}\log_p(\gamma_{\mathfrak{p}_j})\alpha_{\mathfrak{p}_j}([p,F_j])^{-1} = \frac{\log_p(N_j(q_j))}{\mathrm{ord}_p(N_j(q_j))},$$

since $\alpha_{\mathfrak{p}_j}([p,F_j]) = 1$ (by split multiplicative reduction of E at $\mathfrak{p}_j$ with $j \le k$). From this, the desired formula follows from (1.3). □

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