$\mathcal{L}$-INVARIANTS OF CM FIELDS

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1. Lecture 4

Let p be an odd prime. Let M/F be a totally imaginary quadratic extension of the base totally real field F. We study the adjoint square Selmer group when the Galois representation is an induction of a Galois character of $\mathfrak{G}_M := \mathrm{Gal}(M^{(p)}/M)$. Put $\mathfrak{G}_F := \mathrm{Gal}(M^{(p)}/F)$. For simplicity, we assume that $p > 2$ totally splits in $M/\mathbb{Q}$. We relate the Selmer group with a more classical Iwasawa module of a quadratic extension of F, and from the torsion property of the Selmer group already proven, we deduce some (new) torsion property of such classical Iwasawa modules.

1.1. **Ordinary CM fields and their Iwasawa modules.** Let O_M be the integer ring of M. We consider $Z = \varprojlim_n Cl_M(p^n)$ for the ray class group $Cl_M(p^n)$ of M modulo p^n. Let Δ be the maximal torsion subgroup of Z, and put $\Gamma_M = Z/\Delta$, which has a natural action of $\mathrm{Gal}(M/F)$. We split $Z = \Delta \times \Gamma_M$. We define $\Gamma^+ = H^0(\mathrm{Gal}(M/F), \Gamma_M)$ and $\Gamma^- = \Gamma_M/\Gamma^+$. Since $p > 2$, the action of $\mathrm{Gal}(M/F)$ splits the extension $\Gamma^+ \hookrightarrow \Gamma_M \twoheadrightarrow \Gamma^-$, and we have a canonical decomposition $\Gamma_M = \Gamma^+ \times \Gamma^-$. Write $\pi^- : Z \to \Gamma^-$, $\pi^+ : \Gamma_M \to \Gamma^+$ and $\pi_\Delta : Z \to \Delta$ for the three projections. Take a character $\varphi : \Delta \to \overline{\mathbb{Q}}^\times$, and regard it as a character of Z through the projection: $Z \twoheadrightarrow \Delta$.

Let M_∞ be the composite of all $\mathbb{Z}_p$-extensions of M. Then by class field theory, M_∞ is the subfield of the ray class filed of M modulo p^∞ fixed by Δ. Let $\mathbb{Q}_\infty/\mathbb{Q}$ be the cyclotomic $\mathbb{Z}_p$-extension. Let M_∞^{cyc} be the composite $M\mathbb{Q}_\infty/M$. Define M_∞^- (resp. M_∞^+) for the fixed subfield of Γ^- (resp. Γ^+). Since M_∞^{cyc} is abelian over F, we have $M_\infty^{cyc} \subset M_\infty^+$ and a projection $\pi_{cyc} : \Gamma^+ \to \mathrm{Gal}(M_\infty^{cyc}/M) \subset 1 + p\mathbb{Z}_p$. The Leopoldt conjecture for F asserts that π_{cyc} is an isomorphism; in other words, $M_\infty^+ = M_\infty^{cyc}$. The extension M_∞^-/M is called the anticyclotomic tower over M. Thus if the Leopoldt conjecture holds for F, M_∞ is the composite of the cyclotomic $\mathbb{Z}_p$-extension M_∞^{cyc} and the anticyclotomic $\mathbb{Z}_p^{[F:\mathbb{Q}]}$–extension M_∞^-.

To introduce Iwasawa modules for the multiple $\mathbb{Z}_p$-extensions $M_\infty^?/M$, we fix a CM type Σ, which is a set of embeddings of M into $\overline{\mathbb{Q}}$ such that $I_M = \Sigma \sqcup \Sigma c$ for the

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generator c of $\mathrm{Gal}(M/F)$. Over $\mathbb{C}$, an abelian variety with complex multiplication by M has $\mathbb{C}$-points isomorphic to $\mathbb{C}^\Sigma/\Sigma(\mathfrak{a})$ for a lattice $\mathfrak{a}$ in M (see [ACM] 5.2), where $\Sigma(\mathfrak{a}) = \{(\sigma(a))_{\sigma\in\Sigma} \in \mathbb{C}^\Sigma | a \in \mathfrak{a}\}$. By composing i_p, we write Σ_p for the set of p-adic places induced by $i_p \circ \sigma$ for $\sigma \in \Sigma$. We assume

$$\Sigma_p \cap \Sigma_p c = \emptyset. \tag{spt}$$

This is to guarantee the abelian variety of CM type Σ to have ordinary good reduction modulo p (whose Galois representation is hence ordinary at all $\mathfrak{p}|p$).

Writing $M(p^\infty)$ for the ray class field over M modulo p^∞, we identify Z with $\mathrm{Gal}(M(p^\infty)/M)$ via the Artin reciprocity law. Fix a character φ of Δ. We then define M_Δ by the fixed field of Γ in $M(p^\infty)$; so, $\mathrm{Gal}(M_\Delta/M) = \Delta$.

Since φ is a character of Δ, φ factors through $\mathrm{Gal}(M^?_\infty M_\Delta/M)$ for ? indicating one of $+, -, cyc$ or "nothing". When nothing is attached, it refers to the object for the full multiple $\mathbb{Z}_p$-extension M_∞. Let $L^?_\infty/M^?_\infty M_\Delta$ be the maximal p–abelian extension unramified outside Σ_p. Each $\gamma \in \mathrm{Gal}(L_\infty/M)$ acts on the normal subgroup $X^? = \mathrm{Gal}(L^?_\infty/M^?_\infty M_\Delta)$ continuously by conjugation, and by the commutativity of $X^?$, this action factors through $\mathrm{Gal}(M_\Delta M^?_\infty/M)$. Then we look into the compact p-profinite $\Gamma^?$–module: $X^?[\varphi] = X^? \otimes_{\mathbb{Z}_p[\Delta],\varphi} W$, where $\Gamma^? = \mathrm{Gal}(M^?_\infty/M)$. We study when $X^?[\varphi]$ is a torsion Iwasawa module over $\Lambda^? = W[[\Gamma^?]]$. The module $X^?[\varphi]$ is generally expected to be torsion of finite type over $\Lambda^?$ for the naturally defined multiple $\mathbb{Z}_p$-extensions $M^?_\infty$.

The torsion property of $X^{cyc}[\varphi]$ over Λ^{cyc} is classically known (e.g., [HT2] Theorem 1.2.2). This implies

Theorem 1.1. *The modules $X[\varphi], X^+[\varphi]$ and $X^{cyc}[\varphi]$ are torsion modules over the corresponding Iwasawa algebra Λ, Λ^+ and Λ^{cyc}, respectively.*

We refer this result to [HT2] Theorem 1.2.2 (which was originally due to R. Greenberg). We study the anticyclotomic Iwasawa module $X^-[\varphi]$ over Λ^- from our new view point of Galois deformation theory. As is well known, $X^-[\varphi]$ is a Λ^--module of finite type, and under mild assumptions (including anticyclotomy of φ), we will prove the torsion property of $X^-[\varphi]$ in Theorem 1.3.

The Σ-Leopoldt conjecture for abelian extensions of M is almost equivalent to the torsion property of $X^-[\varphi]$ over Λ^- for all possible φ (see [HT2] Theorem 1.2.2). Here, for an abelian extension L/M with integer ring O_L, the Σ-Leopoldt conjecture asserts the closure $\overline{O_L^\times}$ of $O_L^\times$ in $L_\Sigma = \prod_{\mathfrak{p}\in\Sigma_p} L_\mathfrak{p}$ satisfies

$$\dim_\mathbb{Q}(O_L^\times \otimes_\mathbb{Z} \mathbb{Q}) = \dim_{\mathbb{Q}_p}(\overline{O_L^\times} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p).$$

If $X^-[\varphi]$ is a torsion Λ^--module, we can think of the characteristic element $\mathcal{F}^-(\varphi) \in \Lambda^-$ of the module $X^-[\varphi]$. The anticyclotomic main conjecture (cf. [HT] Conjecture 2.2) predicts the identity (up to units) of $\mathcal{F}^-(\varphi)$ and the projection of (the φ-branch of) the Katz p-adic L-function (constructed in [K] and [HT1]) under π^-.

1.2. **Anticyclotomic Iwasawa modules.** A character ψ of Δ is called *anticyclotomic* if $\psi(c\sigma c^{-1}) = \psi^{-1}(\sigma)$ for a complex conjugation $c \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$. Fix an algebraic closure $\overline{F}$ of F. Regarding φ as a Galois character, we define $\varphi^-(\sigma) = \varphi(c\sigma c^{-1}\sigma^{-1})$ for $\sigma \in \mathrm{Gal}(\overline{F}/M)$. Then $\psi := \varphi^-$ is anticyclotomic.

We define a Galois character $\widetilde{\varphi} : \mathfrak{G}_F \to W[[\Gamma^-]]$ by $\widetilde{\varphi}(\sigma) = \varphi(\sigma)(\sigma|_{M_\infty^-})^{1/2}$, where $(\sigma|_{M_\infty^-})^{1/2}$ is the unique square root of $(\sigma|_{M_\infty^-})$ in Γ^- and $(\sigma|_{M_\infty^-}) \in \Gamma^-$ is regarded as a group element in $\Gamma^- \subset W[[\Gamma^-]]$. Note that $\widetilde{\varphi}^-(\sigma) = \widetilde{\varphi}(c\sigma c^{-1}\sigma^{-1}) = \psi(\sigma)\sigma|_{M_\infty^-}$. Then we consider $\mathrm{Ind}_M^F(\widetilde{\varphi}) : \mathrm{Gal}(\overline{F}/F) \to GL_2(W[[\Gamma^-]])$. We write $\alpha_{M/F}$ for the quadratic character of $\mathrm{Gal}(\overline{F}/F)$ identifying $\mathrm{Gal}(M/F)$ with $\{\pm 1\}$.

Lemma 1.2. *We have*

(1) $\det(\mathrm{Ind}_M^F \chi) = \alpha_{M/F}\chi|_{F_{\mathbb{A}}^\times}$ *and* $\mathrm{Tr}(\mathrm{Ind}_M^F \chi(Frob_{\mathfrak{l}})) = \sum_{\mathfrak{b}\subset O_M, N_{M/F}(\mathfrak{b})=\mathfrak{l}} \chi(\mathfrak{b})$ *for a prime* $\mathfrak{l}$ *of* F *unramified for* $\mathrm{Ind}_M^F \chi$, *identifying a character* χ *of* $\mathrm{Gal}(\overline{F}/M)$ *with a character of* $M_{\mathbb{A}^{(\infty)}}^\times/M^\times$ *by the Artin symbol,*

(2) $Ad(\mathrm{Ind}_M^F(\widetilde{\varphi})) \cong \alpha_{M/F} \oplus \mathrm{Ind}_M^F(\widetilde{\varphi}^-)$ *as* $\mathfrak{G}_F$*-modules.*

Since $\mathrm{Ind}_M^F(\widetilde{\varphi})|_{\mathfrak{G}_M} = \widetilde{\varphi} \oplus \widetilde{\varphi}_c$ with $\widetilde{\varphi}_c(\sigma) = \widetilde{\varphi}(c\sigma c^{-1})$, we define $\mathcal{F}_{\mathfrak{p}}^+ \mathrm{Ind}_M^F \widetilde{\varphi} = \widetilde{\varphi}$ for $\mathfrak{p} \in \Sigma_p$. In Lecture 2, we have already defined $\mathcal{F}_{\mathfrak{p}}^{\pm} Ad(\mathrm{Ind}_M^F \widetilde{\varphi})$ and the Selmer group $\mathrm{Sel}_F(Ad(\mathrm{Ind}_M^F \widetilde{\varphi}) \otimes_{\mathbb{Z}_p} (W[[\Gamma^-]])^*)$. Since the image of $\mathcal{F}_{\mathfrak{p}}^+(Ad(\mathrm{Ind}_M^F \widetilde{\varphi}))$ in $\alpha_{M/F}$ is trivial in the above decomposition in Lemma 1.2 and the image of $\mathcal{F}_{\mathfrak{p}}^+(Ad(\mathrm{Ind}_M^F \widetilde{\varphi}))$ is given by $\mathcal{F}_{\mathfrak{p}}^+(\mathrm{Ind}_M^F(\widetilde{\varphi}^-))$, we get (cf. [HMI] Exercise 1.12 and Corollary 3.81)

$$
\begin{aligned}
&\mathrm{Sel}_F(Ad(\mathrm{Ind}_M^F(\widetilde{\varphi}) \otimes_{W[[\Gamma^-]]} (W[[\Gamma^-]])^*) \\
&\qquad= \mathrm{Sel}_F\left(\alpha_{M/F} \otimes_{\mathbb{Z}_p} (W[[\Gamma^-]])^*\right) \oplus \mathrm{Sel}_F(\mathrm{Ind}_M^F(\widetilde{\varphi}^-) \otimes_{W[[\Gamma^-]]} (W[[\Gamma^-]])^*) \\
&\qquad= \mathrm{Hom}(Cl_M^- \otimes_{\mathbb{Z}} W[[\Gamma^-]], \mathbb{Q}_p/\mathbb{Z}_p) \oplus \mathrm{Sel}_M((\widetilde{\varphi}^-) \otimes_{W[[\Gamma^-]]} W[[\Gamma^-]]^*),
\end{aligned}
$$

where Cl_M^- is the quotient of CL_M by the image of Cl_F (the order of Cl_M^- is equal to the order of the $\alpha_{M/F}$-eigenspace of Cl_M up to a power of 2). By the definition of the Selmer group, we note that

$$
\mathrm{Sel}_M(\widetilde{\varphi}^- \otimes_{W[[\Gamma^-]]} (W[[\Gamma^-]])^*) \cong \mathrm{Hom}(X^-[\varphi^-], \mathbb{Q}_p/\mathbb{Z}_p), \tag{1.1}
$$

which shows

Theorem 1.3. *Let the notation be as above. Then we have*

$$
\mathrm{Sel}_F^*(Ad(\mathrm{Ind}_M^F(\widetilde{\varphi}))) \cong \left(Cl_M^- \otimes_{\mathbb{Z}} W[[\Gamma^-]]\right) \oplus X^-[\varphi^-]
$$

as $W[[\Gamma^-]]$*-modules. Moreover* $X^-[\varphi^-]$ *is a torsion* $W[[\Gamma^-]]$*-module without exceptional zero if* $\psi := \varphi^-$ *satisfies the following conditions:*

(at1) *The character* ψ *has order prime to* p.

(at2) *The local character* $\psi_{\mathfrak{P}}$ *is non-trivial for all* $\mathfrak{P} \in \Sigma_p$.

(at3) *The restriction* ψ^* *of* ψ *to* $\mathrm{Gal}(\overline{F}/M^*)$ *for the composite* M^* *of* M *and the unique quadratic extension inside* $F[\mu_p]$ *is non-trivial.*

The first assertion follows from the argument given as above. The torsion property follows from the theorem of Taylor–Wiles and Fujiwara and the propositions in the following appendix. In [HMI] Theorem 5.33, it is checked under assumptions milder than (at1–3) imply the assumption of the theorem of Taylor–Wiles and Fujiwara.

1.3. **The $\mathcal{L}$-invariant of CM fields.** Consider the universal couple $(\mathcal{R}_F, \varrho)$ deforming $\rho_F = \mathrm{Ind}_M^F \varphi \mod \mathfrak{m}_W$ among W-deformations ρ_A into $GL_2(A)$ for proartinian W-algebras A with residue field $W/\mathfrak{m}_W$ satisfying the following conditions

- (W1) unramified outside p;
- (W2) $\rho_A|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/F_\mathfrak{p})} \cong \left(\begin{smallmatrix} * & * \\ 0 & \alpha_{A,\mathfrak{p}} \end{smallmatrix}\right)$ with $\alpha_{A,\mathfrak{p}} \equiv \alpha_\mathfrak{p} \mod \mathfrak{m}_A$ and $\alpha_{A,\mathfrak{p}}\alpha_\mathfrak{p}^{-1}|_{I_\mathfrak{p}}$ factoring through $\mathrm{Gal}(F_\mathfrak{p}[\mu_{p^\infty}]/F_\mathfrak{p})$;
- (W3) $\det(\rho_A) = \det \rho$;
- (W4) $\rho_A \equiv \overline{\rho} \mod \mathfrak{m}_A$,

We know $\mathcal{R}_F \cong \mathbb{T}$ by Fujiwara (see Appendix). Since $\dim \mathrm{Spf}(W[[\Gamma^-]]) = \dim \mathrm{Spf}(\mathbb{T})$, $\mathrm{Spf}(W[[\Gamma^-]])$ gives an irreducible component of $\mathrm{Spf}(\mathcal{R}_F)$. Write $\mathbb{I} = W[[\Gamma^-]]$ simply. Let $\pi_\mathbb{I} : \mathbb{T} = \mathcal{R}_F \twoheadrightarrow \mathbb{I}$ be the projection (which factors through π^{cyc}). We would like to compute the $\mathcal{L}$-invariant of the component $\mathbb{I}$. Thus we need to compute $\mathbf{a}(p_\mathfrak{p}) = \pi_\mathbb{I}(U(p_\mathfrak{p}))$. The following fact follows from the fact $\mathrm{Ind}_M^F \phi|_{\mathfrak{G}_M} = \phi \oplus \phi^c$.

Lemma 1.4. *Let the notation be as above. Then we have $\mathbf{a}(p_\mathfrak{p}) = \widetilde{\varphi}([p_\mathfrak{P}, M_\mathfrak{P}])$ for the prime factor $\mathfrak{P} \in \Sigma_p^c$ of $\mathfrak{p}$.*

Define the character $\boldsymbol{\kappa} : \mathrm{Gal}(\overline{F}/M) \to (\Lambda^-)^\times$ by $\boldsymbol{\kappa}(\sigma) = (\sigma|_{M_\infty^-})^{1/2}$. Then $\widetilde{\varphi} = \lambda\boldsymbol{\kappa}$, and we write $\boldsymbol{\kappa}_\mathbb{I} = \pi_\mathbb{I} \circ \boldsymbol{\kappa} : \mathrm{Gal}(\overline{F}/M) \to \mathbb{I}^\times$. Then, $\boldsymbol{\kappa}_\mathbb{I}$ restricted to the inertia group $I_\mathfrak{P}$ at $\mathfrak{P}$ factors through the projection: $I_\mathfrak{P} \to \mathrm{Gal}(\mathbb{Q}_p[\mu_{p^\infty}]/\mathbb{Q}_p) \cong \mathbb{Z}_p^\times$. Since the $W[[\Gamma_F]]$-algebra structure of $\mathbb{I}$ is induced by the nearly ordinary character of $\mathrm{Ind}_M^F \boldsymbol{\kappa}_\mathbb{I}$ (restricted to the inertia group $I_\mathfrak{P}$), for $u_\mathfrak{P} \in O_{M,\mathfrak{P}}^\times$ ($\mathfrak{P} \in \Sigma_p^c$), we have

$$\boldsymbol{\kappa}_\mathbb{I}([u_\mathfrak{P}, M_\mathfrak{P}]) = (1 + X_\mathfrak{p})^{-\log_p(N_\mathfrak{p}(u_\mathfrak{P}))/\log_p(\gamma_\mathfrak{p})}, \tag{1.2}$$

where $\mathfrak{p} = \mathfrak{P} \cap O$, $N_\mathfrak{p} : M_\mathfrak{P} = F_\mathfrak{p} \to \mathbb{Q}_p$ is the norm map and $\gamma_\mathfrak{p}$ is the generator of $\Gamma_\mathfrak{p} := (1 + p\mathbb{Z}_p) \cap N_\mathfrak{p}(O_\mathfrak{p}^\times)$. Choose an element $\alpha(\mathfrak{P}) \in M$ so that $\mathfrak{P}^h = (\alpha(\mathfrak{P}))$ for each $\mathfrak{P} \in \Sigma_p^c$, where $h = |Cl_M|$ (the class number of M). Then $p_\mathfrak{P}^h = u_\mathfrak{P}\alpha(\mathfrak{P})_\mathfrak{P}^{e(\mathfrak{p})}$ with $u_\mathfrak{P} \in O_{M,\mathfrak{P}}^\times$ for the absolute ramification index $e(\mathfrak{p})$ of $\mathfrak{p}$ (which is the absolute ramification index of $\mathfrak{P}$ also). Regarding $\boldsymbol{\kappa}_\mathbb{I}$ as a character of $M_{\mathbb{A}^{(\infty)}}^\times/M^\times$ by class field theory, we have $\boldsymbol{\kappa}_\mathbb{I}(\alpha(\mathfrak{P})) = 1 = \boldsymbol{\kappa}_\mathbb{I}(\alpha(\mathfrak{P})_\mathfrak{l})$ with the $\mathfrak{l}$-component $\alpha(\mathfrak{P})_\mathfrak{l} \in M_\mathfrak{l}^\times$ for any prime $\mathfrak{l}$ outside p, because $\boldsymbol{\kappa}_\mathbb{I}(O_\mathfrak{l}^\times) = 1$ and $\alpha(\mathfrak{P}) \in M^\times$. Then we have

$$\boldsymbol{\kappa}_\mathbb{I}(p_\mathfrak{P}^h) = \boldsymbol{\kappa}_\mathbb{I}(p_\mathfrak{P}^h\alpha(\mathfrak{P})^{-e(\mathfrak{p})}) = \boldsymbol{\kappa}_\mathbb{I}(u_\mathfrak{P}) \prod_{\mathfrak{P}'|p, \mathfrak{P}' \neq \mathfrak{P}} \boldsymbol{\kappa}_\mathbb{I}(\alpha(\mathfrak{P})_{\mathfrak{P}'}^{-e(\mathfrak{p})}),$$

where $\alpha(\mathfrak{P})_{\mathfrak{P}'}$ is the $\mathfrak{P}'$-component of $\alpha(\mathfrak{P}) \in M^\times \subset M_{\mathbb{A}}^\times$. By (1.2), we get

$$\boldsymbol{\kappa}_{\mathbb{I}}(p_{\mathfrak{P}}^h) = (1+X_{\mathfrak{p}})^{\frac{\log_p(N_{\mathfrak{p}}(\alpha(\mathfrak{P})^{-e(\mathfrak{p})c}u_{\mathfrak{p}}^{-1}))}{\log_p(\gamma_{\mathfrak{p}})}} \prod_{\mathfrak{P}' \in \Sigma_p^c - \{\mathfrak{P}\}} (1+X_{\mathfrak{p}'})^{\frac{e(\mathfrak{p})\log_p(N_{\mathfrak{p}'}(\alpha(\mathfrak{P})_{\mathfrak{P}'}^{1-c}))}{\log_p(\gamma_{\mathfrak{p}'})}},$$

where $\mathfrak{p}' = \mathfrak{P}' \cap O$. Here $\log_p$ is the Iwasawa p-adic logarithm defined over $\overline{\mathbb{Q}}_p^\times$ characterized by $\log_p(p) = 0$. In particular, we have

$$\log_p(N_{\mathfrak{p}}(u_{\mathfrak{P}})) = \log(N_{\mathfrak{p}}(p_{\mathfrak{P}}^h \alpha(\mathfrak{P})_{\mathfrak{P}}^{-e(\mathfrak{p})})) = -e(\mathfrak{p})\log_p(N_{\mathfrak{p}}(\alpha(\mathfrak{P})_{\mathfrak{P}})).$$

Thus we have

Lemma 1.5. *Let the notation be as above. Then we have, for primes $\mathfrak{P}' \in \Sigma_p$ and $\mathfrak{p}' = \mathfrak{P}' \cap O$,*

$$\frac{\partial \boldsymbol{\kappa}(p_{\mathfrak{P}})}{\partial X_{\mathfrak{p}'}} = \frac{e(\mathfrak{p})\log_p(N_{\mathfrak{p}'}(\alpha(\mathfrak{P})_{\mathfrak{P}'}^{(1-c)}))}{h\log_p(\gamma_{\mathfrak{p}'})} \boldsymbol{\kappa}(p_{\mathfrak{p}})(1+X_{\mathfrak{p}'})^{-1}.$$

We have $\mathbf{a}(p_{\mathfrak{p}}) = c_{\mathfrak{p}}\boldsymbol{\kappa}(p_{\mathfrak{p}})$ for a nonzero constant $c_{\mathfrak{p}} \in W^\times$, because the nearly ordinary character of $\mathrm{Ind}_M^F \widetilde{\varphi}$ is $\boldsymbol{\kappa}$ times a character of $D_{\mathfrak{P}}$ with values in $W^\times$. We do not need to pay much attention to the constant $c_{\mathfrak{p}}$, because the formula of the $\mathcal{L}$-invariant only involve

$$(\prod_{\mathfrak{p}|p} \mathbf{a}(p_{\mathfrak{p}})^{-1}\delta_{\mathfrak{p}}([\gamma_{\mathfrak{p}}, F_{\mathfrak{p}}])) \det\left(\left(\frac{\partial \mathbf{a}(p_{\mathfrak{p}})}{\partial X_{\mathfrak{p}'}}\right)_{\mathfrak{p},\mathfrak{p}'}\right)$$

in which the constant $c_{\mathfrak{p}}$ cancels out. Specializing the above formula to the locally cyclotomic point P, we get

Theorem 1.6. *Let the notation and the assumption be as above and as in* Theorem 1.3, *including* (at1–4). *Then we have, for any specialization $\widetilde{\varphi}_P$ of $\widetilde{\varphi}$ modulo a locally cyclotomic point $P \in \mathrm{Spf}(\mathbb{I})(W)$,*

$$\mathcal{L}(Ad(\mathrm{Ind}_M^F \widetilde{\varphi}_P)) = \det\left(\left(\log_p(N_{\mathfrak{p}'}(\alpha(\mathfrak{P})_{\mathfrak{P}'}^{(1-c)}))\right)_{\mathfrak{P},\mathfrak{P}' \in \Sigma_p^c}\right) \prod_{\mathfrak{p}|p} \frac{e(\mathfrak{p})}{h},$$

where $\mathfrak{p} = O \cap \mathfrak{P}$ and $\mathfrak{p}' = O \cap \mathfrak{P}'$

By Lemma 1.2 (2) and Theorem 1.3, we see

$$\mathcal{L}(Ad(\mathrm{Ind}_M^F \widetilde{\varphi}_P)) = \mathcal{L}(\alpha_{M/F}),$$

and this is the reason for the independence of $\mathcal{L}(Ad(\mathrm{Ind}_M^F \widetilde{\varphi}_P))$ on the choice of the locally cyclotomic points P. If $F = \mathbb{Q}$, we have $\alpha(\mathfrak{P})\alpha(\mathfrak{P})^c = p^h$ and hence $\log_p(\alpha(\mathfrak{P})) = -\log_p(\alpha(\mathfrak{P})^c)$. Thus $\log_p(\alpha(\mathfrak{P})^{1-c}) = 2\log_p(\alpha(\mathfrak{P}))$, and therefore the above formula coincides with the classical analytic $\mathcal{L}$-invariant formula for $\alpha_{M/F}$ of Ferrero–Greenberg.

For a given ordinary CM type (M, Σ_p), we can choose ψ satisfying the assumptions of Theorems 1.3 and 1.6. Then through the above process, we can compute $\mathcal{L}(\alpha_{M/F})$ as follows:

Corollary 1.7. *Suppose that M/F is an ordinary CM-quadratic extension of M satisfying* (spt). *Choose a p-ordinary CM-type Σ of M. Then the $\mathcal{L}$-invariant $\mathcal{L}(\alpha_{M/F})$ of Greenberg for the quadratic Galois character $\alpha_{M/F} = \left(\frac{M/F}{\cdot}\right)$ is given by*

$$\det\left(\left(\log_p(N_{\mathfrak{p}'}(\alpha(\mathfrak{P})^{(1-c)}_{\mathfrak{P}'}))\right)_{\mathfrak{P},\mathfrak{P}'\in\Sigma_p^c}\right)\prod_{\mathfrak{p}|p}\frac{e(\mathfrak{p})}{h},$$

where h is the class number of M, $\mathfrak{p}' = \mathfrak{P}' \cap O$ and $\alpha(\mathfrak{P})$ is a generator of $\mathfrak{P} \in \Sigma_p^c$. If the prime p does not split in $F/\mathbb{Q}$, the $\mathcal{L}$-invariant of $\alpha_{M/F}$ does not vanish.

A regulator similar to the above determinant was introduced long ago in [FeG] (3.8) in the context of (classical) cyclotomic Iwasawa's theory.

2. Appendix:Differential and adjoint square Selmer group

2.1. **Adjoint square Selmer groups and differentials.** Recall the **universal** nearly ordinary deformation $\boldsymbol{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to GL_2(R)$ over K with the pro-Artinian local universal K-algebra R. This means that for any Artinian local K-algebra A with maximal ideal $\mathfrak{m}_A$ and any Galois representation $\rho_A : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to GL_2(A)$ such that

(K1) unramified outside p;
(K2) $\rho_A|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/F_\mathfrak{p})} \cong \left(\begin{smallmatrix} * & * \\ 0 & \alpha_{A,\mathfrak{p}} \end{smallmatrix}\right)$ with $\alpha_{A,\mathfrak{p}} \equiv \alpha_\mathfrak{p} \mod \mathfrak{m}_A$;
(K3) $\det(\rho_A) = \det\rho$;
(K4) $\rho_A \equiv \rho \mod \mathfrak{m}_A$,

there exists a unique K-algebra homomorphism $\varphi : R \to A$ such that $\varphi \circ \boldsymbol{\rho} \cong \rho_A$. We write $\Phi_K(A)$ the collection of the isomorphism classes of the deformations ρ_A.

Let $\overline{\rho} = (\rho \mod \mathfrak{m}_W)$, and consider a similar deformation changing base ring from K to W. Then we have a universal couple $(\mathcal{R}, \varrho)$ as long as (ai$_F$) $\overline{\rho}$ is absolutely irreducible and (ds) $\overline{\rho}^{ss}$ is not scalar-values over $D_\mathfrak{p}$ for all $\mathfrak{p}|p$ (these assumptions we always assume). This means that for any pro-Artinian local W-algebra A with $A/\mathfrak{m}_A = W/\mathfrak{m}_W = \mathbb{F}$ for the maximal ideal $\mathfrak{m}_A$ and any Galois representation $\rho_A : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to GL_2(A)$ such that

(W1) unramified outside p;
(W2) $\rho_A|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/F_\mathfrak{p})} \cong \left(\begin{smallmatrix} * & * \\ 0 & \alpha_{A,\mathfrak{p}} \end{smallmatrix}\right)$ with $\alpha_{A,\mathfrak{p}} \equiv \alpha_\mathfrak{p} \mod \mathfrak{m}_A$;
(W3) $\det(\rho_A) = \det\rho$;
(W4) $\rho_A \equiv \overline{\rho} \mod \mathfrak{m}_A$,

there exists a unique W-algebra homomorphism $\varphi : \mathcal{R} \to A$ such that $\varphi \circ \varrho \cong \rho_A$. We write $\Phi(A)$ the collection of the isomorphism classes of this finer deformations ρ_A. Thus $\Phi(A) \cong \mathrm{Hom}_{W-alg}(R, A)$.

Let $\widetilde{\rho} \in \Phi(A)$ acting on $\widetilde{L}$. Define

$$\widetilde{T} = \left\{ \phi \in \mathrm{End}_A(\widetilde{L}) \middle| \mathrm{Tr}(\phi) = 0 \right\}. \tag{2.1}$$

We let $\sigma \in \mathfrak{G}_F$ act on $v \in \widetilde{T}$ by conjugation $v \mapsto \widetilde{\rho}(\sigma) v \widetilde{\rho}(\sigma)^{-1}$. As before, $\widetilde{T}$ has the following three step filtration stable under $D_{\mathfrak{p}}$ for each prime ideal $\mathfrak{p}|p$ of F:

$$\widetilde{T} \supset \mathcal{F}^-_{\mathfrak{p}} \widetilde{T} \supset \mathcal{F}^+_{\mathfrak{p}} \widetilde{T} \supset \{0\}. \tag{2.2}$$

Let $\mathbb{Z}_p^* = \mathbb{Q}_p/\mathbb{Z}_p = \mathrm{Hom}(\mathbb{Z}_p . \mathbb{Q}_p/\mathbb{Z}_p)$ and $A^* = \mathrm{Hom}(A, \mathbb{Q}_p/\mathbb{Z}_p)$. We thus have $\mathrm{Sel}_F(V/T) = \mathrm{Sel}_F(T \otimes \mathbb{Z}_p^*)$ for $V/T := T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$, and we also have

$$\mathrm{Sel}_F(Ad(\widetilde{\rho})) = \mathrm{Sel}_F(\widetilde{T} \otimes_A A^*) = \mathrm{Ker}(H^1(\mathfrak{G}, \widetilde{T} \otimes_A A^*) \to \prod_{\mathfrak{p}|p} H^1(I_{\mathfrak{p}}, \frac{\widetilde{T} \otimes_A A^*}{\mathcal{F}^+_{\mathfrak{p}} \widetilde{T} \otimes_A A^*}))$$

for $\widetilde{\rho} \in \Phi(A)$ and the inertia subgroup $I_{\mathfrak{p}} \subset \mathfrak{G}$. Note that $D_{\mathfrak{p}}$ acts trivially on $\mathcal{F}^-_{\mathfrak{p}} V / \mathcal{F}^+_{\mathfrak{p}} V$. We often indicate this fact by writing $\mathcal{F}^-_{\mathfrak{p}} V / \mathcal{F}^+_{\mathfrak{p}} V \cong K$ as $D_{\mathfrak{p}}$-modules.

Proposition 2.1. *Suppose that Φ has a universal couple $(\mathcal{R}_F, \varrho_F)$. Then the Pontryagin dual $\mathrm{Sel}^*_F(V/T)$ is canonically isomorphic to the module of 1-differentials $\Omega_{\mathcal{R}_F/W[[\Gamma_F]]} \otimes_{\mathcal{R}_F, \varphi} W$, where $\varphi : \mathcal{R}_F \to W$ is the W-algebra homomorphism such that $\rho \cong \varphi \circ \varrho_F$. More generally, for any $\widetilde{T} \in \Phi(A)$, we have*

$$\mathrm{Sel}^*_F(\widetilde{T} \otimes_A A^*) = \mathrm{Hom}(\mathrm{Sel}_F(\widetilde{T} \otimes_A A^*), \mathbb{Z}_p^*) \cong \Omega_{\mathcal{R}_F/W[[\Gamma_F]]} \otimes_{\mathcal{R}_F, \phi} A,$$

where $\phi : \mathcal{R}_F \to A$ is the W-algebra homomorphism such that $\widetilde{\rho} \cong \phi \circ \varrho_F$.

This proposition is from [MFG] Theorem 5.14. Here *Kähler 1-differentials* are supposed to be continuous with respect to the profinite topology.

Here is a sketch of a **proof**: Write simply $(\mathcal{R}, \varrho)$ for $(\mathcal{R}_F, \varrho_F)$. Let $\Phi = \Phi^{n.ord,\nu}$; so, $\Phi(A) \cong \mathrm{Hom}_{W\text{-alg}}(\mathcal{R}, A)$. For simplicity, we assume that X be a profinite $\mathcal{R}$-module, (in general, we take an inductive limit of such modules). Then $\mathcal{R}[X]$ is an object in CL_W. We consider the W-algebra homomorphism $\xi : \mathcal{R} \to \mathcal{R}[X]$ with ξ mod $X = \mathrm{id}$. Then we can write $\xi(r) = r \oplus d_\xi(r)$ with $d_\xi(r) \in X$. By the above definition of the product, we get $d_\xi(rr') = r d_\xi(r') + r' d_\xi(r)$ and $d_\xi(W) = 0$. Thus d_ξ is a W-derivation, i.e., $d_\xi \in Der_W(\mathcal{R}, X)$. For any derivation $d : \mathcal{R} \to X$ over W,

$r \mapsto r \oplus d(r)$ is obviously a W-algebra homomorphism, and we get

$$\begin{aligned}(2.3)\quad &\left\{\widetilde{\rho} \in \Phi(\mathcal{R}[X]) \middle| \widetilde{\rho} \mod X = \varrho\right\} / \approx_X \\ &\qquad\cong \left\{\widetilde{\rho} \in \Phi(\mathcal{R}[X]) \middle| \widetilde{\rho} \mod X \approx \varrho\right\} / \approx \\ &\qquad\cong \left\{\xi \in \mathrm{Hom}_{W\text{-alg}}(\mathcal{R}, \mathcal{R}[X]) \middle| \xi \mod X = \mathrm{id}\right\} \\ &\qquad\cong Der_W(\mathcal{R}, X) \cong \mathrm{Hom}_{\mathcal{R}}(\Omega_{\mathcal{R}/W}, X),\end{aligned}$$

where "$\approx_X$" is conjugation under $(1 \oplus M_n(X)) \cap GL_2(\mathcal{R}[X])$, and "$\approx$" is conjugation by elements in $GL_2(\mathcal{R}[X])$.

Let $\widetilde{\rho}$ be the deformation in the left-hand side of (2.3). Then we may write $\widetilde{\rho}(\sigma) = \varrho(\sigma) \oplus u'(\sigma)$ (here $u'(\sigma)$ is a "derivative" of $\widetilde{\rho}(\sigma)$). We see

$$\varrho(\sigma\tau) \oplus u'(\sigma\tau) = (\varrho(\sigma) \oplus u'(\sigma))(\varrho(\tau) \oplus u'(\tau)) = \varrho(\sigma\tau) \oplus (\varrho(\sigma)u'(\tau) + u'(\sigma)\varrho(\tau)).$$

Define $u(\sigma) = u'(\sigma)\varrho(\sigma)^{-1}$, which is a cocycle with values on $M_2(X)$ by the above formula. Since $\det \widetilde{\rho} = \det \rho = \det \varrho$, $x(\sigma) = \widetilde{\rho}(\sigma)\varrho(\sigma)^{-1}$ has values in $SL_2(\mathcal{R}[X])$, u has values in $Ad(X) = L(Ad(\varrho)) \otimes_{\mathcal{R}} X$. Hence $u : \mathfrak{G}_F^S \to Ad(X)$ is a 1-cocycle. It is a straightforward computation to see the injectivity of the map:

$$\left\{\widetilde{\rho} \in \Phi(\mathcal{R}[X]) \middle| \widetilde{\rho} \mod X \approx \varrho\right\} / \approx_X \hookrightarrow H^1(\mathfrak{G}_F^S, Ad(X))$$

given by $\widetilde{\rho} \mapsto [u]$. We put $\mathcal{F}_{\mathfrak{p}}^{\pm}(Ad(X)) = \mathcal{F}_{\mathfrak{p}}^{\pm} L(Ad(\varrho)) \otimes_{\mathcal{R}} X$. Since $\varrho|_{I_{\mathfrak{p}}}$ is upper-triangular (up to conjugation), we have $u|_{I_{\mathfrak{p}}}$ has values in $\mathcal{F}_{\mathfrak{p}}^{-} Ad(X)$.

If further we insist on $d_\xi(W[[\Gamma_F]]) = 0$, since $W[[\Gamma_F]]$-algebra structure is given by $\boldsymbol{\delta}_{\mathfrak{p}}\alpha_{\mathfrak{p}}^{-1}$ which is the character of lower right corner of ϱ (restricted to $I_{\mathfrak{p}}$) this means the corresponding cocycle $u|_{I_{\mathfrak{p}}}$ has values in $\{\left(\begin{smallmatrix} * & * \\ 0 & 0 \end{smallmatrix}\right)\}$. Since $\mathrm{Tr}(u) = 0$, we conclude $u|_{I_{\mathfrak{p}}} \in \mathcal{F}_{\mathfrak{p}}^{+} Ad(X)$. If $\widetilde{L}$ is finite, we may tale $X = \widetilde{L}$, and this gives the desired isomorphism, because $\widetilde{T} \otimes_A A^* = Ad(\widetilde{L})$. If $\widetilde{L}$ is not finite, then $\widetilde{L} \otimes_A A^*$ can be written as a union of $Ad(X)$ for finite X, and by taking the inductive limit, we get the assertion.. □

If we replace $\mathcal{F}^{+}\widetilde{?}$ in the definition of the Selmer group by $\mathcal{F}^{-}\widetilde{?}$, we get the "minus" Selmer group $\mathrm{Sel}_F^{-}(?)$, and by the same argument

$$\mathrm{Sel}_F^{-}(\widetilde{T} \otimes_A A^*)^* \cong \Omega_{\mathcal{R}_F/W} \otimes_{\mathcal{R}} A.$$

We can apply the above argument to $(R, \boldsymbol{\rho})$. If $\rho \in \Phi(W)$, we have a unique $P \in \mathrm{Spf}(\mathcal{R}_F)(W)$ such that $\varrho \mod P = \rho$. Then R is canonically isomorphic to the P-adic completion-localization $\widehat{\mathcal{R}}_P$ of $\mathcal{R}$ at P and $\boldsymbol{\rho} : \mathfrak{G}_F \xrightarrow{\varrho} GL_2(\mathcal{R}) \to GL_2(\widehat{\mathcal{R}}_P) = GL_2(R)$. Thus we get

Corollary 2.2. *We have $\Omega_{R/K} \otimes_{R,\varphi_\rho} K \cong \mathrm{Sel}_F^{-}(V)$ which is isomorphic to $\bigoplus_{\mathfrak{p}|p} K dX_{\mathfrak{p}}$ under the conjecture: $R \cong K[[X_{\mathfrak{p}}]]$.*

Under the conjecture, the Selmer group $\mathrm{Sel}_F^-(V)$ is exactly $\mathbb{H} \subset H^1(\mathfrak{G}, V)$ discussed in the second lecture, and the restriction map takes $\mathbb{H} = \mathrm{Sel}_F^-(V)$ into $\mathrm{Sel}_{F_\infty}(V)$ as we have seen. Recall Greenberg's formula for a base $a_\mathfrak{p}$ of $\mathbb{H} \overset{\mathrm{Res}}{\cong} \mathcal{H}$:

$$\mathcal{L}(Ad(\rho)) = \det\left((a_\mathfrak{p}([p, F_{\mathfrak{p}'}])_{\mathfrak{p},\mathfrak{p}'|p} \left(\log_p(\gamma_\mathfrak{p})^{-1} a_\mathfrak{p}([\gamma_{\mathfrak{p}'}, F_{\mathfrak{p}'}])\right)_{\mathfrak{p},\mathfrak{p}'|p}\right)^{-1} \right).$$

Then by the above corollary, putting $c_\mathfrak{p} = \frac{\partial \boldsymbol{\rho}}{\partial X_\mathfrak{p}} \boldsymbol{\rho}^{-1}\Big|_{X=0}$, $\{c_\mathfrak{p}\}_{\mathfrak{p}|p}$ is a basis of $\mathbb{H}$. Then writing $c_\mathfrak{p} \sim \left(\begin{smallmatrix} -a_\mathfrak{p} & * \\ 0 & a_\mathfrak{p} \end{smallmatrix}\right)$, we can compute the above formula. Note that $a_\mathfrak{p} = \boldsymbol{\delta}_\mathfrak{p}^{-1} \frac{\partial \boldsymbol{\rho}}{\partial \boldsymbol{\delta}_\mathfrak{p}}\Big|_{X=0}$, and $\boldsymbol{\delta}_\mathfrak{p}([\gamma_\mathfrak{p}, F_\mathfrak{p}]) = (1 + X_\mathfrak{p})$ and $\boldsymbol{\delta}_\mathfrak{p}([p_\mathfrak{p}, F_\mathfrak{p}]) = U(p_\mathfrak{p})$. From this we get the formula we stated in the first lecture.

We add the following condition to the deformations $\widetilde{L}$ satisfying (W1–4) to make the universal ring small enough to prove $\mathrm{Sel}_F(V/T)$ is finite (and $\mathrm{Sel}_F(V) = 0$). Let Σ_p be the set of all prime factors of p in O. Fix a pair of integers $(\kappa_{1,\mathfrak{p}}, \kappa_{2,\mathfrak{p}})$ for each $\mathfrak{p} \in \Sigma_p$, and write κ for the tuple $(\kappa_{1,\mathfrak{p}}, \kappa_{2,\mathfrak{p}})_\mathfrak{p}$. We assume that $[\kappa] = \kappa_{1,\mathfrak{p}} + \kappa_{2,\mathfrak{p}}$ is independent of $\mathfrak{p} \in \Sigma_p$. As an extra condition, we now consider

(W5) On $\widetilde{T}/\mathcal{F}_\mathfrak{p}^+\widetilde{T}$, $\mathrm{Gal}(F_\mathfrak{p}^{ur}[\mu_{p^\infty}]/F_\mathfrak{p}^{ur})$ acts by the character $\mathcal{N}^{\kappa_{1,\mathfrak{p}}}$ for all $\mathfrak{p}|p$, and $\det(\widetilde{T}) = \mathcal{N}^{[\kappa]}$ on an open subgroup of $I_\mathfrak{p}$.

We write $\Phi_\kappa(A)$ for the set of isomorphism classes of deformations $\widetilde{\rho} : \mathfrak{G}_F \to GL_2(A)$ of $\overline{\rho}$ satisfying (W1–5). Under (ai_F) or (ds), we have the universal couple $(R_{\kappa,F}, \varrho_{\kappa,F})$ among the deformations satisfying (W1–5). We call $c \in \mathfrak{G}_F$ a complex conjugation, if c is in the conjugacy class of a complex conjugation in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Conjecture 2.3. *Suppose* (ds) *and* (ai_F) *for* $\overline{\rho}$ *and that* F *is totally real. If* $\det(\rho)(c) = -1$ *for any complex conjugation* c*, the universal ring* $R_{\kappa,F}$ *is free of finite rank over* W*, and* $R_{\kappa,F}$ *is a reduced local complete intersection if* $\kappa_{2,\mathfrak{p}} - \kappa_{1,\mathfrak{p}} \geq 1$ *for all* $\mathfrak{p} \in \Sigma_p$*.*

Here a reduced algebra A free of finite rank over $W[[x_1, \ldots, x_t]]$ is a *local complete intersection* over $R = W[[x_1, \ldots, x_t]]$ if $A \cong R[[T_1, \ldots, T_r]]/(f_1(T), \ldots, f_r(T))$ for r power series $f_i(T)$, where r is the number of variables in $R[[T_1, \ldots, T_r]]$. Though the assertion of $R_{\kappa,F}$ being a local complete intersection is technical, as we will see later, this claim is a key to relating the size of the Selmer group with the corresponding L-value. In the classical setting of Galois representations associated to elliptic modular forms of weight k (in $S_k(\Gamma_1(N))$), we have $\kappa = (0, k-1)$. Thus the condition $\kappa_{2,\mathfrak{p}} - \kappa_{1,\mathfrak{p}} \geq 1$ is equivalent to requiring $k \geq 2$.

Theorem 2.4 (Wiles, Taylor, Fujiwara)**.** *Suppose that the initial representation* ρ *is associated to a Hilbert modular form of* p*-power level (in this case, we call* ρ *modular). If* (ai_M) *holds for* $M = F[\mu_p]$, Conjecture 2.3 *holds.*

See Fujiwara's paper: arXiv.math.NT/0602606. A more general version of this theorem is proven as Theorem 3.67 and Corollary 3.42 in [HMI].

Proposition 2.5. *Assume* Conjecture 2.3. *Then*

(1) $\mathrm{Sel}_F(Ad(\widetilde{\rho}) \otimes_W W^*)$ *is finite for any* $\widetilde{\rho} \in \Phi_k(W)$ *and* $R \cong K[[X_{\mathfrak{p}}]]_{\mathfrak{p}\in\Sigma_p}$ *if* $\kappa_{2,\mathfrak{p}} - \kappa_{1,\mathfrak{p}} \geq 1$ *for all* $\mathfrak{p} \in \Sigma_p$,

(2) $\mathcal{R}_F$ *is a reduced local complete intersection free of finite rank over* $W[[\Gamma_F]]$,

(3) $\mathrm{Sel}^*_F(Ad(\varrho_F) \otimes_{\mathcal{R}_F} \mathcal{R}^*_F)$ *is a torsion* $\mathcal{R}_F$*-module,*

(4) *For an irreducible component* $\mathrm{Spf}(\mathbb{I})$ *of* $\mathrm{Spf}(\mathcal{R}_F)$, *write* $\rho_{\mathbb{I}} = \pi \circ \varrho_F$ *for the projection* $\pi : \mathcal{R}_F \twoheadrightarrow \mathbb{I}$. *Then* $\mathrm{Sel}^*_F(Ad(\rho_{\mathbb{I}}) \otimes_{\mathbb{I}} \mathbb{I}^*)$ *is a torsion* $\mathbb{I}$*-module.*

Proof. If R is reduced and free of finite rank over W, $\Omega_{R/W}$ is a finite module. Thus the first assertion follows. Note that $P_\kappa = \mathrm{Ker}(\kappa : W[[\Gamma_F]] \to W)$ is generated by $((1 + x_{\mathfrak{p}}) - \mathcal{N}(\gamma_{\mathfrak{p}})^{\kappa_{1,\mathfrak{p}}})$ for $\mathfrak{p} \in S$. Thus $\cap_\kappa P_\kappa = \{0\}$. Since $\mathcal{R}_F/P_\kappa\mathcal{R}_F \cong R_{\kappa,F}$ which is free of finite rank s over W, by Nakayama's lemma, $\mathcal{R}_F$ is generated by s elements $r_1, \ldots, r_s$ over $W[[\Gamma_F]]$ which give a basis of $R_{\kappa,F}$ over W. Thus we have a surjective $W[[\Gamma_F]]$-linear map $\iota : W[[\Gamma_F]]^s \to \mathcal{R}_F$ sending $(a_1, \ldots, a_s)$ to $\sum_j a_j r_j$. Taking another κ', we find that $\mathcal{R}_F/P_{\kappa'}\mathcal{R}_F \cong R_{\kappa',F}$ which is free over W; so, it has to be free of rank s over W. Thus $\mathrm{Ker}(\iota) \subset P^s_{\kappa'}$ for all κ'; so, ι has to be an isomorphism. This shows the freeness in the second assertion.

Let C be the set of all $\kappa = (\kappa_{\mathfrak{p}})_{\mathfrak{p}}$ such that $\kappa_{2,\mathfrak{p}} - \kappa_{1,\mathfrak{p}} \geq 1$ for all $\mathfrak{p}$. Then we still have $\bigcap_{\kappa\in C} P_\kappa = \{0\}$. Thus the natural W-algebra homomorphism $\mathcal{R}_F \to \prod_{\kappa\in C} R_{\kappa,F}$ is an injection. The right-hand side is reduced (i.e., no nilpotent radical), and $\mathcal{R}_F$ is reduced.

We write

$$R_\kappa = \mathcal{R}/P_\kappa = \frac{W[[T_1, \ldots, T_r]]}{(\overline{f}_1(T), \ldots, \overline{f}_r(T))}.$$

Write $\overline{t}_j \in \mathfrak{m}_{R_\kappa}$ for the image of T_j in R_κ. Take a lift t_j in $\mathfrak{m}_{\mathcal{R}}$ of $\overline{t}_j$ so that $\overline{t}_j = (t_j \bmod P_\kappa\mathcal{R})$. Define $\varphi : W[[\Gamma_F]][[T_1, \ldots, T_r]] \twoheadrightarrow \mathcal{R}_F$ by $\varphi(f(T_1, \ldots, T_r)) = f(t_1, \ldots, t_r)$. Since $\mathcal{R}_F$ is $W[[\Gamma_F]]$-free, $\mathrm{Ker}(\varphi) \otimes_{W[[\Gamma_F]],\kappa} W = (\overline{f}_1, \ldots, \overline{f}_r)$; so, taking a lift $f_j \in \mathrm{Ker}(\varphi)$ of $\overline{f}_j$, we find $\mathrm{Ker}(\varphi) = (f_1, \ldots, f_r)$ by Nakayama's lemma, and hence $\mathcal{R}_F$ is a local complete intersection over $W[[\Gamma_F]]$.

Since $\mathcal{R}_F$ is reduced and finite over $W[[\Gamma_F]]$, $\Omega_{\mathcal{R}_F/W[[\Gamma_F]]}$ is a torsion $\mathcal{R}_F$-module. From this, the last two assertions follow. Since $R_{\kappa,F} = \mathcal{R}_F/P_\kappa\mathcal{R}_F$ is reduced, $\mathrm{Spf}(\mathcal{R}_F)$ is étale over $\mathrm{Spf}(W[[\Gamma_F]])$ around $\rho = P$; so, $R = \widehat{\mathcal{R}}_P \cong K[[X_{\mathfrak{p}}]]$. This finishes the proof. $\square$

Since $\mathcal{R}_F$ is reduced and free of finite rank over $W[[\Gamma_F]]$, its total quotient ring Q is a product of fields of finite dimension over the field $\mathcal{K}$ of fractions of $W[[\Gamma_F]]$. For simplicity, we assume that $\mathbb{I} = W[[\Gamma_F]]$. In particular, writing $\mathbb{K}$ for the field of fractions of $\mathbb{I}$, we have $Q = \mathbb{K} \oplus X$ for a complementary ring direct summand X. Let $\mathbb{I}'$ be the projection of $\mathcal{R}_F$ to X. Then $\mathrm{Spf}(\mathcal{R}_F) = \mathrm{Spf}(\mathbb{I}) \cup \mathrm{Spf}(\mathbb{I}')$ (and $\mathrm{Spf}(\mathbb{I}')$ is the union of irreducible components other than $\mathrm{Spf}(\mathbb{I})$). We take the intersection $\mathrm{Spf}(C_0) = \mathrm{Spf}(\mathbb{I}) \cap \mathrm{Spf}(\mathbb{I}')$; so, $C_0 = \mathbb{I} \otimes_{\mathcal{R}_F} \mathbb{I}'$, which is a torsion $\mathbb{I}$-module called the *congruence module* of $\mathbb{I}$ (or of $\mathrm{Spf}(\mathbb{I})$). It is easy to see that

$\mathbb{I} \otimes_{\mathcal{R}_F} \mathbb{I}' \cong \mathbb{I}/((\mathbb{K} \oplus 0) \cap \mathcal{R}_F)$ (cf. [H88] 6.3). By the above expression, the $W[[\Gamma_F]]$-freeness tells us that $\mathrm{char}_{\mathbb{I}}(C_0) = (\mathbb{K} \oplus 0) \cap \mathcal{R}_F$ is an intersection of a power of prime divisors (cf. [BCM] 7.4.2). Since $\mathbb{I} = W[[\Gamma_F]]$ is regular, and hence $\mathrm{char}(C_0)$ is a principal ideal generated by $h \in \mathbb{I}$. For this conclusion, we do not need the isomorphism $\mathbb{I} \cong W[[\Gamma_F]] = W[[x_{\mathfrak{p}}]]_{\mathfrak{p}}$ but a milder condition that $\mathbb{I}$ is a Gorenstein ring over $W[[\Gamma_F]]$ is enough (that is, $\mathrm{Hom}_{W[[\Gamma_F]]}(\mathbb{I}, W[[\Gamma_F]]) \cong \mathbb{I}$ as $\mathbb{I}$-modules; see [H88] Theorem 6.8). Note that a local complete intersection over $W[[\Gamma_F]]$ is a Gorenstein ring (e.g., [CRT] Theorem 21.3). Now by a theorem of Tate (e.g., [MFG] 5.3.4),

$$\mathrm{char}(\Omega_{\mathcal{R}_F/W[[\Gamma_F]]} \otimes_{\mathcal{R}_F} \mathbb{I}) = \mathrm{char}(C_0) = (h).$$

We have for any prime ideal $P \in \mathrm{Spf}(\mathbb{I})$ with $\iota : \mathbb{I}/P \cong W$, writing $\rho_P = \iota \circ \rho_{\mathbb{I}} : \mathfrak{G}_F \to GL_2(W)$

$$\mathrm{Sel}^*_F(Ad(\rho_P) \otimes_W W^*) \cong \Omega_{\mathcal{R}_F/W[[\Gamma_F]]} \otimes_{\mathcal{R}_F,P} W \cong \mathrm{Sel}^*_F(Ad(\rho_{\mathbb{I}}) \otimes_{\mathbb{I}} \mathbb{I}^*) \otimes_{\mathbb{I}} \mathbb{I}/P.$$

This shows that if $\mathrm{char}(\mathrm{Sel}^*_F(Ad(\rho_{\mathbb{I}}) \otimes_{\mathbb{I}} \mathbb{I}^*)) = (h)$ for $h \in \mathbb{I}$, we have

$$\mathrm{char}(\mathrm{Sel}^*_F(Ad(\rho_P) \otimes_W W^*)) = (h(P)),$$

where $h(P) = (h \mod P) \in W$. Thus we get

Corollary 2.6. *We have* $|\mathrm{Sel}^*_F(Ad(\rho_P) \otimes_W W^*)| = |h(P)|_p^{-[K:\mathbb{Q}_p]}$ *for all* $P \in \mathrm{Spf}(\mathbb{I})(W)$.

In this corollary, we do not preclude the case where $\mathrm{Sel}^*_F(Ad(\rho_P) \otimes_W W^*)$ is infinite. In such an extreme case, simply $h(P) = 0$ and, hence, $|h(P)|_p^{-[K:\mathbb{Q}_p]} = \infty$.

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