Tilouine's lecture

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$$\mathcal{H}^N = \bigotimes_{\ell \nmid N} \overbrace{\mathbb{Z}[T_{\ell,i}]}^{\mathcal{H}_\ell}{}_{i=0,1,2}$$

eigensystem:

$\theta_f : \mathcal{H}^N \longrightarrow \mathbb{C}$ ring hom.

If f is eigen. $k = (k_1, k_2)$. $k_1 \geq k_2$

$\mathrm{Im}\, \theta_f \subset \mathcal{O}_E$. E number field.

Hecke polynomial

$P_{f,\ell} \in \mathcal{O}_E[X]$.

$$\begin{array}{ccc} \mathcal{H}^N[x] & \ni & P_\ell \\ \theta_f \downarrow & & \updownarrow \\ \mathcal{O}_E[x] & \ni & P_{f,\ell} \end{array}$$

$$U_\ell = M(\mathbb{Z}_\ell)\begin{pmatrix}1 & \\ & \ell\end{pmatrix}M(\mathbb{Z}_\ell) \in \mathbb{Q}[M(\mathbb{Z}_\ell)\backslash M(\mathbb{Q}_\ell)/M(\mathbb{Z}_\ell)]$$

Hecke Frobenius

$$\begin{array}{c} \mathbb{Q}[M(\mathbb{Z}_\ell)\backslash M(\mathbb{Q}_\ell)/M(\mathbb{Z}_\ell)] \\ W_G \big\uparrow \text{Satake} \\ \mathcal{H}_\ell \subset \mathbb{Q}[G(\mathbb{Z}_\ell)\backslash G(\mathbb{Q}_\ell)/G(\mathbb{Z}_\ell)] \end{array}$$

$$M = \left\{ \begin{pmatrix} A & 0 \\ 0 & {}^t\!A^{-1} \end{pmatrix} \in Sp_4 \right\}$$

$$P_\ell = \mathrm{Irr}(X; U_\ell, \mathbb{Q}[G(\mathbb{Z}_\ell)\backslash G(\mathbb{Q}_p)/G(\mathbb{Z}_\ell)])$$
$$= X^4 - T_{\ell,1} X^3 + \cdots + \ell^6 T_{\ell,0}^2$$

Andrianov, 3.3.35.

$P_{f,\ell} \longrightarrow$ (prime-to-N-part of) degree four automorphic L-function of f

$$L^{(N)}(f, s) = \prod_{\ell \nmid N} P_{f,\ell}(\ell^{-s})^{-1}$$

Fix a prime p. $f \in S_k(\Gamma)$. k cohomological

type Γ_1

$\mathbb{Q} \xhookrightarrow{\iota_p} \overline{\mathbb{Q}}_p$

Theorem (R. Taylor, Laumon, Weissauer, Ast 302)

$\exists\, F$, p-adic field, $F \supset \iota_p(E)$

$\exists\, \rho_{f,p} : G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_4(F)$

semisimple, unramified outside Np,

$$\forall \ell \nmid Np, \quad \det(X 1_4 - \rho_{f,p}(\mathrm{Fr}_\ell)) = P_{f,\ell}(X)$$

It is conjectural that if f is not "particular"

(red. ind. repr. $\leftrightarrow$ corresp) CAP or weakly endoscopic

then:

(Sympl) $\rho_{f,p}$ takes values in $GSp_4(F)$ s.t.

$$\underset{\text{similitude}}{\nu} \circ \rho_{f,\ell} = \chi^{-(k_1+k_2-3)}\, \omega_f^{\mathrm{gal}}$$

$\omega_f : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$. $\quad f|\langle a\rangle = \omega_f(a) f$

$\langle a \rangle \equiv \begin{pmatrix} a^{-1} & & & \\ & a^{-1} & & \\ & & a & \\ & & & a \end{pmatrix} (N)$

$\omega_f^{\mathrm{gal}}(\mathrm{Fr}_\ell) = \omega_f(\ell)$

Aside:

type Γ_1: $\quad \Gamma \ni \gamma \equiv \begin{pmatrix} 1 & \\ 0 & * \\ 0 & \\ 0 & \end{pmatrix} (N)$

using $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$:

e.g. $\Gamma_{1,B}(N) = \left\{ \gamma \equiv \begin{pmatrix} 1 & * & \multicolumn{2}{c}{X} \\ & 1 & & \\ \multicolumn{2}{c}{0} & 1 & 0 \\ & & * & 1 \end{pmatrix} (\mathrm{mod}\ N) \right\}$

$$\begin{tikzcd}[column sep=small] & \Gamma_{0,P} \arrow[dr, phantom, "\subset"] & \\ \Gamma_{0,B} \arrow[ur, phantom, "\subset"] \arrow[dr, phantom, "\subset"] & & \Gamma_\Phi \\ & \Gamma_{0,Q} \arrow[ur, phantom, "\subset"] & \end{tikzcd}$$

$$\Gamma_{1,B} \supset \Gamma_{1,P} \longleftrightarrow \begin{pmatrix} 1 & * & * & * \\ & 1 & * & \\ & & 1 & \\ & & * & 1 \end{pmatrix}$$

$$\Gamma_{1,B} \supset \Gamma_{1,Q} \longleftrightarrow \begin{pmatrix} 1 & & & * \\ & & & \\ & & & \\ & & & 1 \end{pmatrix}$$

$$\begin{tikzcd}[column sep=small] & G(\mathbb{Z}_\ell) & \\ \Pi_P \arrow[ur, phantom, "\subset"] & & \Pi_Q \arrow[ul, phantom, "\supset"] \end{tikzcd}$$

$G'(\mathbb{Z}_\ell) = \Pi^{\mathrm{param}}$

(Irr) $\rho_{f,\ell}$ is abs. irred.

Rmk. If f has "global mult." 1, then (Sympl) holds.

If the motivic weight of f, $k_1+k_2-3 < \frac{p+1}{2}$, then (Irr) holds.

Conj. If f is not particular, then it has global mult. 1.

Assume from now on that $\rho_{f,\ell}$ is symplectic and abs. irreducible.

In particular, $\nu \circ \rho_{f,p}(c) = -1$.

$$\rho_{f,p}(c) \sim \begin{pmatrix} \boxed{\begin{matrix} 1 & \\ & 1 \end{matrix}} & \\ & \boxed{\begin{matrix} -1 & \\ & -1 \end{matrix}} \end{pmatrix}$$

isotropic

Conjecture k finite field of char. p.

$$\bar{\rho} : G_{\mathbb{Q}} \longrightarrow GSp_4(k)$$

continuous

If $\bar{\rho}$ abs. irred. and odd, then $\exists f \in S_k(\Gamma)$, k cohom

s.t. $\bar{\rho}_{f,p} = \bar{\rho}$ (Has to be made more precise.)

(Relation between conductor and level is not clear.)

Conjecture (Generalized modularity conj.)

$$\rho : G_{\mathbb{Q}} \longrightarrow GSp_4(F).$$

cont. abs. irred. geometric

1) If HT weights are regular, $a < b < c < d$, $a+d=b+c$

then $\exists$ k cohom. $\exists$ f eigen. in $S_k(\Gamma')$

s.t. $\rho = \rho_{f,p} \otimes \chi^a$

$b-a = k_2 - 2$

$c-a = k_1 - 1$

$d-a = k_1 + k_2 - 3$

2) relatively singular HT weights. (a, a, b, b). $a<b$.

$\exists f \in S_{2, b-a+1}(\Gamma')$. $\rho = \rho_{f,p} \otimes \chi^a$

3) totally singular, (a, a, a, a).

No holomorphic Siegel modular form should realize ρ. (or no coherently cohomological SMF ...)

Start with (in absence of Serre's conj.)

$\rho : G_{\mathbb{Q}} \to GSp_4(F)$

Assume :

$\bar{\rho} = \bar{\rho}_{f_0, p}$. $f_0 \in S_{k_0}(\Gamma)$, k_0 cohom.

After a long list of assumptions on ρ and f_0, one concludes $\rho = \rho_{g,p}$

with g s.t.

1) If $HT(\rho)$ are regular, $0<b<c<d$,

$g \in S_k(\Gamma \cap \Gamma_{1,B}(p))$

k determined by b, c, d as above.

2) If $HT(\rho)$ is relatively singular, $(0, 0, 1, 1)$

$g \in S^{\dagger}_{2,2}(\Gamma)$ (overconvergent)

3) If $HT(\rho)$ is totally singular or $b-a>1$,

g is generalized p-adic modular form (Katz)

(W.L.O.G. assume $a=0$)

Proved by Tilouine

assumptions:

$K_0 = (k_{0,1}, k_{0,2})$

$p-1 > k_{0,1} + k_{0,2} - 3$ (cf. $p-1 > k-1$ in GL_2 case)

$p \nmid N = \text{level}(\Gamma)$

ρ nearly ordinary (for which Hida's theory works)

minimality of $\bar{\rho}_{f,p}$, ρ $\quad \rho(F_\ell) = \begin{pmatrix} 1 & 1 & 1 & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & & 1 & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$, etc.

(I don't understand the details)

large image of $\bar{\rho} = \bar{\rho}_{f,p}$ in $GSp_4(K)$. $Sp_4(K) \subset \mathrm{Im}\,g$.

$$\rho|_{I_p} = \begin{pmatrix} \chi_1 & * & & * \\ & \chi_2 & & \\ & & \chi_3 & \\ 0 & & * & \chi_4 \end{pmatrix} \qquad \chi_i = \chi^{-w_i}\theta_i$$

Errata, precisions and complements on Lecture 2, J. Tilouine

1) The (twisted) Satake homomorphism

$$\mathbb{Q}\left[G(\mathbb{Z}_\ell)\backslash G(\mathbb{Q}_\ell)/G(\mathbb{Z}_\ell)\right] \hookrightarrow \mathbb{Q}\left[M(\mathbb{Z}_\ell)\backslash M(\mathbb{Q}_\ell)/M(\mathbb{Z}_\ell)\right]$$

defines a non-Galois extension of degree four, generated by the "Hecke Frobenius" $U_\ell = [M(\mathbb{Z}_\ell)(^{1}\,{}_{\ell_2})M(\mathbb{Z}_\ell)]$.

Then, P_ℓ is defined as $\mathrm{Irr}(X; U_\ell, \mathbb{Q}[G(\mathbb{Z}_\ell)\backslash G(\mathbb{Q}_\ell)/G(\mathbb{Z}_\ell)])$

2) After the existence theorem for the degree four Galois representation $\rho_{f,p} : G_{\mathbb{Q}} \to GL_4(F)$ associated to an arbitrary cusp eigenform of cohomological weight, one should list the following remarks:

1) It is conjectured that this representation is always symplectic, with similitude fact $\chi^{-(k_1+k_2-3)}\omega_f^{gal}$ where ω_f is the Dirichlet character defined as the finite part of the central character of f.

2) If f is not "particular", one conjectures $\rho_{f,p}$ absolutely irreducible.

3) "particular" means <u>either</u> CAP (= Saito-Kurokawa lift) in which case $\rho_{f,p} = \psi \oplus \rho_{g,p} \oplus \psi'$ where g is a cusp eigenf. on $GL(2,\mathbb{Q})$ and ψ, ψ' are 1-dim. repres.

<u>or</u> f is a (weak) endoscopic lift from $GL(2)\times GL(2)$ in which case $\rho_{f,p} = \rho_{g_1,p} \oplus (\rho_{g_2,p} \otimes \psi)$

where ψ is a 1-dim. rep.

let $f \longleftrightarrow \pi^\infty \otimes \pi_\infty^{hol} = \pi$ cusp. aut. repres of $GSp_4(\mathbb{A})$.
let K be a congruence subgroup of $GSp_4(\hat{\mathbb{Z}})$ corresponding to $\Gamma \subset Sp_4(\mathbb{Z})$.

Theorem: If f is not "particular", Θ_f occurs in $H^3(Y_\Gamma, V_a(\mathbb{C}))$. Moreover

if $m_{hol}(\pi^\infty) = \mathrm{mult}(\pi^\infty \otimes \pi_\infty^{hol})$

and $m_{wh}(\pi^\infty) = \mathrm{mult}(\pi^\infty \otimes \pi_\infty^{wh})$

then

$$4.\ \dim\left(H^3(Y_\Gamma, V_a(\mathbb{C}))\left[\Theta_f\right]\right) = 2\left(m_{hol}(\pi^\infty) + m_{wh}(\pi^\infty)\right).\dim(\pi^\infty)^K$$

· Def: π^∞ has multiplicity one if $m_{hol}(\pi^\infty) = 1$ or $m_{wh}(\pi^\infty) = 1$.

π' is weakly equivalent to π if for almost all prime $\pi'_\ell \sim \pi_\ell$.

Proposition If f is not particular and π^∞ is weakly ~~has multi~~ equivalent ~~to~~ to a multiplicity one representation, then

$$m_{hol}(\pi^\infty) = m_{wh}(\pi^\infty) = 1$$

and $\rho_{f,p}$ is symplectic.

Conjecture: If f is not particular, π^∞ is weakly equivalent to a multiplicity one representation

Theorem If f is not particular and π^∞ is weakly equivalent to a multiplicity one rep., then $\rho_{f,p}$ is Hodge-Tate with weights $0,\ k_2-2,\ k_1-1,\ k_1+k_2-3$