

# Tiloume's lecture

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Hida J. Inst. Math. Jussieu 2, 2001.

$$\mathcal{H}^N = \bigotimes_{\mathbb{Z}[T_{\ell,i}]}^{\mathcal{H}_\ell} \quad i=0,1,2$$

eigensystem:

$$\theta_f : \mathcal{H}^N \rightarrow \mathbb{C} \text{ ring hom.}$$

If  $f$  is eigen,  $k = (k_1, k_2)$ ,  $k_1 \geq k_2$

$\text{Im } \theta_f \subset \mathcal{O}_E$ ,  $E$  number field.

Hedcke polynomial

$$P_{f,\ell} \in \mathcal{O}_E[X]$$

$$\mathcal{H}^N[X] \ni P_\ell$$

$$\theta_f \downarrow \quad \bar{\phantom{P}}$$

$$\mathcal{O}_E[X] \ni P_{f,\ell}$$

$$U_\ell = M(\mathbb{Z}) \begin{pmatrix} 1 & \\ & \ell \end{pmatrix} M(\mathbb{Z}) \in \mathbb{Q} [ M(\mathbb{Z}_\ell) \backslash M(\mathbb{Q}_\ell) / M(\mathbb{Z}_\ell) ]$$

Hedcke Frabinius

$W_G \uparrow$  Satake

$$\mathcal{H}_\ell \subset \mathbb{Q} [ G(\mathbb{Z}_\ell) \backslash G(\mathbb{Q}_\ell) / G(\mathbb{Z}_\ell) ]$$

$$M = \left\{ \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \in \text{Sp}_4 \right\}$$

$$P_\ell = \text{Irr}(X; U_\ell, \mathbb{Q} [ G(\mathbb{Z}_\ell) \backslash G(\mathbb{Q}_\ell) / G(\mathbb{Z}_\ell) ])$$

$$= X^4 - T_{\ell,1} X^3 + \dots + \ell^6 T_{\ell,0}^2$$

$$1 \text{st } G(\ell) : X^3 - T_{\ell,1} X^2 + \dots$$

Andriaman, 3. 3. 35.

$P_{f,l} \rightsquigarrow$  (prime-to- $N$ -part of)  
degree four automorphic  
L-function of  $f$

$$L^{(N)}(f, s) = \prod_{l \nmid N} P_{f,l}(l^{-s})^{-1}$$

Fix a prime  $p$ ,  $f \in S_{16}(\Gamma)$ ,  $K$  cohomological  
 $\mathbb{Q} \hookrightarrow \bar{\mathbb{Q}}_p$   $\uparrow$  type  $\Gamma$

Theorem (R. Taylor, Lawman, Wiles, Ast 302)

$\exists F$ ,  $p$ -adic field,  $F \supset \mathbb{Q}_p(E)$

$\exists \rho_{l,p}: G_{\mathbb{Q}} = \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \rightarrow GL_4(F)$

semisimple, unramified outside  $N_p$

$$\forall l \nmid N_p, \text{dec}(X^2 - \rho_{l,p}(F_{rel})) = P_{f,l}(X)$$

It is conjectured that if  $l$  is not "particular"

(red. mod  $l$   $\rightarrow$   $\text{Comp}$ ) CAP or  
weakly endoscopic

then:

(Sympl)  $\rho_{l,p}$  takes values in  $GSp_4(F)$  s.t.

smaller  $\rho_{l,p} = \chi^{-(k/2-3)} \omega_f^{\text{gal}}$

$$\omega_f: (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$$

$$\omega_f^{\text{gal}}(F_{rel}) = \omega_f(l)$$

$$f(a) = \omega_f(a) f$$

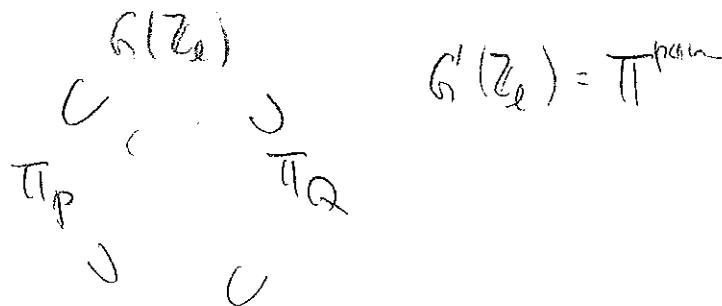
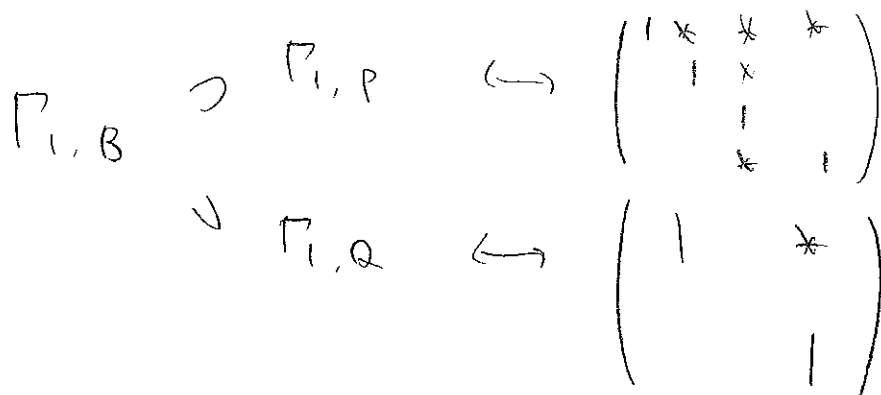
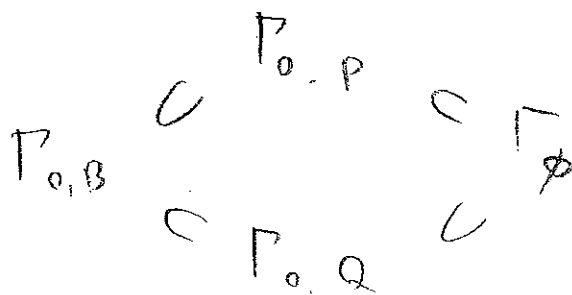
$$\langle a \rangle = \begin{pmatrix} a & & & \\ & a & & \\ & & a & \\ & & & a \end{pmatrix} (N)$$

Aside:

type  $\Gamma_1$ :  $\Gamma \ni \gamma \equiv \begin{pmatrix} 1 & \\ 0 & * \end{pmatrix} (N)$

w/ing  $\begin{pmatrix} 1 & \\ -1 & \end{pmatrix}$ :

e.g.  $\Gamma_{1,B}(N) = \left\{ \gamma \equiv \begin{pmatrix} 1 & * & X \\ 0 & 1 & 0 \\ & * & 1 \end{pmatrix} \pmod{N} \right\}$



(Inv)  $p \neq e$  is abs. inv.

Prp. If  $f$  has "global mult" 1, then (Sympl) holds.

If the motivic weight of  $f$ ,  $k_1 + k_2 - 3 < \frac{p+1}{2}$ , then (Inv) holds.

Conj. If  $f$  is not particular, then  
it has global mult. 1.

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Assume from now on that  $P_{f, \ell}$  is symplectic  
and abs. irreducible.

In particular,  $\forall c \in P_{f, p}(c) = -1$ .

$$P_{f, p}(c) \sim \begin{pmatrix} \square & & & \\ & \square & & \\ & & \square & \\ & & & \square \end{pmatrix} \leftarrow \text{isotropic}$$

Conjecture  $k$  finite field of char.  $p$ .

$$\bar{\rho} = G_{\mathbb{Q}} \longrightarrow \mathrm{GSp}_4(k)$$

(continuous)

If  $\bar{\rho}$  abs. irred. and odd, then  $\exists f \in S_k(\Gamma)$ ,  $k$  coharm.

s.t.  $\bar{\rho}_{f, p} = \bar{\rho}$  (Has to be made more precise.)

(Relation between conductor and level is not clear.)

Conjecture (Generalized modularity conj.)

$$\rho = G_{\mathbb{Q}} \longrightarrow \mathrm{GSp}_4(F)$$

cont. abs. irred. geometric

1) If HT weights are regular,  $a < b < c < d$ ,  $a+d=b+c$ ,  
then  $\exists k$  coharm.  $\exists f$  eigen. in  $S_k(\Gamma')$

s.t.  $\rho = \rho_{f, p} \otimes \chi^a$

$$b-a = k_1 - 2$$

$$c-a = k_1 - 1$$

$$d-a = k_1 + k_2 - 3$$

2) relatively singular HT weights  $(a, a, b, b), a < b$ .

$$\exists f \in S_{2, b-a+1}(\Gamma'). \quad \rho = \rho_{f, p} \otimes \chi^a$$

3) totally singular,  $(a, a, a, a)$ .

No holomorphic Siegel modular form should realize  $\rho$ . (or no coherently cohomological SMF --)

Start with (in absence of Serre's conj-)

$$\rho = \rho_{\mathbb{Q}} \rightarrow \mathrm{GSp}_4(F)$$

Assume:

$$\bar{\rho} = \bar{\rho}_{f_0, p}, \quad f_0 \in S_k(\Gamma), \quad k_0 \text{ cohorn.}$$

After a long list of assumptions on  $\rho$  and  $f_0$ , one concludes  $\rho = \rho_{g, p}$

with  $g$  s.t.

1) If HT( $\rho$ ) are regular,  $0 < b < c < d$ ,

$$g \in S_k(\Gamma \cap \Gamma_{2, B}(p))$$

$k$  determined by  $b, c, d$  as above.

2) If HT( $\rho$ ) is relatively singular,  $(0, 0, 1, 1)$

$$g \in S_{2, 2}^+(\Gamma) \text{ (overconvergent)}$$

3) If HT( $\rho$ ) is totally singular, or  $b-a > 1$ ,

$g$  is generalized p-adic modular form (Katz)

(W.L.O.G. assume  $a=0$ )

Proved by Tilman

assumptions =

$$K_0 = (K_{0,1}, K_{0,2})$$

$$p-1 > k_{0,1} + k_{0,2} - 3 \quad (\text{c.f. } p-1 > k-1 \text{ in } GL_2 \text{ case})$$

$$p \nmid N = \text{level}(\Gamma)$$

$\rho$  nearly ordinary (for which Hida's theory works)

minimality of  $\bar{\rho}|_{F, p}$ ,  $\rho$

$$\rho(F_0) = \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}, \text{ etc.}$$

(I don't understand the second)

large image of  $\bar{\rho} = \bar{\rho}|_{F, p}$  in  $GS_{p^2}(K)$ ,  $Sp_4(K) \subset \text{Im}$ .

$$\rho|_{\mathbb{Z}_p} = \begin{pmatrix} \chi_1 & * & * \\ & \chi_2 & * \\ 0 & * & \chi_3 \\ & * & \chi_4 \end{pmatrix}$$

$$\chi_i = \chi^{-w_i} \theta_i$$

Errata, precisions and complements on Lecture 2, J. Tilouine

1) The (twisted) Satake homomorphism

$$\mathbb{Q} [G(\mathbb{Z}_\ell) \backslash G(\mathbb{Q}_\ell) / G(\mathbb{Z}_\ell)] \hookrightarrow \mathbb{Q} [M(\mathbb{Z}_\ell) \backslash M(\mathbb{Q}_\ell) / M(\mathbb{Z}_\ell)]$$

defines a non-Galois extension of degree four, generated by the "Hecke Frobenius"  $U_\ell = [M(\mathbb{Z}_\ell) \begin{pmatrix} 1 & \\ & \ell \end{pmatrix} M(\mathbb{Z}_\ell)]$ .

Then,  $P_\ell$  is defined as  $\text{Irr}(X; U_\ell; \mathbb{Q} [G(\mathbb{Z}_\ell) \backslash G(\mathbb{Q}_\ell) / G(\mathbb{Z}_\ell)])$

2) After the existence theorem for the degree four

Galois representation  $\rho_{f,p} : G_{\mathbb{Q}} \rightarrow GL_4(F)$  associated to an arbitrary cusp eigenform of cohomological weight, one should list the following remarks:

1) It is conjectured that this representation is always symplectic, with similitude fact  $\chi^{-(k_1+k_2-3)} \omega_f^{\text{gal}}$  where  $\omega_f$  is the Dirichlet character defined as the finite part of the central character of  $f$ .

2) If  $f$  is not "particular", one conjectures  $\rho_{f,p}$  absolutely irreducible.

3) "particular" means either CAP (= Saito-Kurokawa lift)

in which case  $\rho_{f,p} = \psi \oplus \rho_{g,p} \oplus \psi'$  where  $g$  is a cusp eigenf. on  $GL(2, \mathbb{Q})$  and  $\psi, \psi'$  are 1-dim. repres.

or  $f$  is a (weak) endoscopic lift from  $GL(2) \times GL(2)$

$$\rho_{f,p} = \rho_{g_1,p} \oplus (\rho_{g_2,p} \otimes \psi)$$

where  $\psi$  is a 1-dim. rep.

let  $f \leftrightarrow \pi^\infty \otimes \pi_\infty^{\text{hol}} = \pi$  cusp. aut. repres of  $GSp_4(\mathbb{H})$ .

let  $K$  be a congruence subgroup of  $GSp_4(\hat{\mathbb{Z}})$  corresponding to  $\Gamma \subset Sp_4(\mathbb{Z})$ .

Theorem: If  $f$  is not "particular",  $\theta_f$  occurs in  $H^3(Y_\Gamma, V_a(\mathbb{C}))$ . Moreover

$$\text{if } m_{\text{hol}}(\pi^\infty) = \text{mult}(\pi^\infty \otimes \pi_\infty^{\text{hol}})$$

$$\text{and } m_{\text{wh}}(\pi^\infty) = \text{mult}(\pi^\infty \otimes \pi_\infty^{\text{wh}})$$

then

$$4 \cdot \dim(H^3(Y_\Gamma, V_a(\mathbb{C}))[\theta_f]) = 2(m_{\text{hol}}(\pi^\infty) + m_{\text{wh}}(\pi^\infty)) \cdot \dim(\pi^\infty)^{\vee}$$

Def:  $\pi^\infty$  has multiplicity one if  $m_{\text{hol}}(\pi^\infty) = 1$  or  $m_{\text{wh}}(\pi^\infty) = 1$ .

$\pi'$  is weakly equivalent to  $\pi$  if for almost all primes

$$\pi'_l \sim \pi_l.$$

Proposition If  $f$  is not particular and  $\pi^\infty$  is weakly equivalent to a multiplicity one representation, then

$$m_{\text{hol}}(\pi^\infty) = m_{\text{wh}}(\pi^\infty) = 1$$

and  $\rho_{f,p}$  is symplectic.

Conjecture: If  $f$  is not particular,  $\pi^\infty$  is weakly equivalent to a multiplicity one representation

Theorem If  $f$  is not particular and  $\pi^\infty$  is weakly equivalent to a multiplicity one rep., then  $\rho_{f,p}$  is Hodge-Tate with weights  $0, k_2 - 2, k_1 - 1, k_1 + k_2 - 3$