Tilouine's lecture

$\bar\rho : G_{\mathbb{Q}} \to GSp_4(\mathbb{F})$

$\bar\rho \equiv \rho_{f_0,p} (\mathfrak{P})$, $f_0 \in S_{k_0}(\Gamma)$

$p \nmid N$, $p-1 > k_{0,1} + k_{0,2} - 3$ + same conditions (last time)

$HT(\rho) \rightsquigarrow k = (k_1, k_2)$

$\exists\, g$ s.t. $\rho = \rho_{g,p}$

1) k is coh., $g \in S_k(\Gamma \cap \Gamma_{B,1}(p))$ ($k \equiv k_0\ (p-1)$ or not)

2) $k = (2,2)$, $g \in S^{+}_{(2,2)}(\Gamma)$

3) if not, g is generalized p-adic

Ingredients

1) joint work with A Genestier, $k = k_0$

2) Hida theories

$$S_k(\Gamma_{B,1}(p^\infty)) \xrightarrow[\cong]{\text{a Hec}} H^3_!(Y_{\Gamma_{B,1}(p^\infty)}, V_{\underline{a}}(\mathbb{C})) \qquad k = \underline{a} + (3,3)$$

$$\Gamma_{B,1}(p^\infty) = \Gamma \cap \{\gamma \in Sp_4(\mathbb{Z}),\ \gamma \equiv \begin{pmatrix} 1 & * & & * \\ & 1 & & \\ & & 1 & \\ 0 & & * & 1 \end{pmatrix} (p^n)\}$$

$h_k = \mathrm{Im}(\mathcal{H}^{Np} \to \mathrm{End}(\ ))^\wedge$

$S_k(\Gamma_{B,1}(p^\infty))^\wedge_{/\mathbb{Z}_p}$ $\quad H^3_!(\Gamma_{B,1}(p^\infty), V_{\underline{a}}(\mathbb{Z}_p))^\wedge$ ← grp. coh., but actually on $Y_{\Gamma_{B,1}(p^\infty)}$

h_k

then ord. part:

geom. is very nasty

$eS_k(\Gamma_{B,1}(p^\infty))^\wedge_{/\mathbb{Z}_p}$ $\quad eH^3_!(\Gamma_B(p^\infty), V_{\underline{a}}(\mathbb{Z}_p))^\wedge$

ring of func. in the Igusa tower

nasty ord

$eV_{m,n}$ $\quad N(\mathbb{Z}_p)$ $\quad \mathbb{V}_B \circlearrowleft h^{\mathrm{hn}}$

$$R \cong T$$

univ. defor. ring for nearly ord. $\bar{\rho}$

same tricky form of Nakayama

both finite flat

$$T = h^{no}_{\mathfrak{m}} \qquad \mathfrak{m} = \ker\left(\mathcal{H}^{N_p} \xrightarrow{\theta_{f_0}} \mathcal{O}_F \to \mathcal{O}_F/(\varpi)\right)$$

$$\Lambda = \mathcal{O}_F [\![T(\mathbb{Z}_p)_p]\!]$$

known w/ Genestier $\quad R_{K_0} = T_{K_0} \quad G\ \mathbb{V}_B / P_{K_0}\mathbb{V}_B \qquad \begin{pmatrix} a & & & \\ & b & & \\ & & \nu a^{-1} & \\ & & & \nu b^{-1} \end{pmatrix}$

then specialize in wt $(2,2)$

$$\rho_{0,0,1,1} = \rho_g \sim \text{overconvergent}$$

<u>Conjecture</u> If $A/\mathbb{Q}$ abelian surface, irreducible, then $\exists\, g \in S_{2,2}(\Gamma)$, eigen. such that $\rho^\vee_{A,p} = \rho_{g,p}$.

(generalization of Taniyama-Shimura)

<u>Thm</u>: If $A/\mathbb{Q}$ abelian surface

$\forall \ell \neq p$. A has purely toric red. at ℓ or good reduction

p: potential good ordinary

$\rho^\vee_{A,p} \equiv \rho_{f_0,K_0}(\beta)$
?

$\Rightarrow \exists\, g \in S^+_{2,2}(\Gamma)$, $\rho^\vee_{A,p} = \rho_{g,p}$

GL_2

$\Gamma(N) \subset \Gamma \subset SL_2(\mathbb{Z})$
torsion free

$E = \Gamma\backslash\mathfrak{h}_1 \times \mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}$

$\downarrow$

$Y_\Gamma = \Gamma\backslash\mathfrak{h}_1$

$\gamma(z,v) = (\gamma(z), j(\gamma,z)v)$

$(z,v)(a,b) = (z, v+za+b)$

$\mathrm{Lie}(E/Y) = \Gamma\backslash\mathfrak{h}_1 \times \mathbb{C}$

$\underline{\omega} = \Omega^1_{E/Y} = \Gamma\backslash\mathfrak{h}_1 \times \mathbb{C}$

w/ action: $\gamma(z,v) = (\gamma(z), j(\gamma,z)v)$

$k \in \mathbb{Z}$ $\quad \underline{\omega}^{\otimes k}$

$\updownarrow$

$\Gamma \longmapsto \mathcal{O}^*_{\mathfrak{h}_1}$

$\gamma \mapsto (z \mapsto j(\gamma,z)^k)$

GSp_4

$\Gamma(N) \subset \Gamma \subset Sp_4(\mathbb{Z})$

$A = \Gamma\backslash\mathfrak{h}_2 \times \mathbb{C}^2/\mathbb{Z}^2\oplus\mathbb{Z}^2$

$\downarrow$

$Y_\Gamma = \Gamma\backslash\mathfrak{h}_2$

$\gamma(z,v) = (\gamma(z), {}^t j(\gamma,z)^{-1} v)$

$(z,v)(a,b) = (z, v+za+b)$

$\mathrm{Lie}(A/Y) = \Gamma\backslash\mathfrak{h}_2 \times \mathbb{C}^2$

$\underline{\omega} = \Omega^1_{A/Y_\Gamma} = \Gamma\backslash\mathfrak{h}_2 \times \mathbb{C}^2$

$\gamma(z,v) = (\gamma(z), j(\gamma,z)v)$

$\underline{\omega}^k = \Gamma\backslash\mathfrak{h}_2 \times W_k$

$k = (k_1,k_2),\ k_1 \geq k_2$ $\qquad \rho_k : GL_2 \circlearrowright \mathrm{Sym}^{k_1-k_2} \otimes \det^{k_2}(\mathbb{C}^2)$

$\updownarrow$

$\gamma \mapsto (z \mapsto \rho_k(j(\gamma,z)))$

$\underline{\omega} \longleftrightarrow j(\gamma,z)$, $k=(1,0)$

$k = (k,k)$, $\quad W_k = (\Lambda^2\mathbb{C}^2)^{\otimes k}$

p-adically will make sense for any k

$\because$ $\underline{\omega}$ is trivial on the ord. locus

	GL_2	GSp_4
$/\mathbb{C}$	$\begin{array}{ccc} E & \hookrightarrow & \mathcal{G} \\ \downarrow & & \downarrow \uparrow e \\ Y_\Gamma & \hookrightarrow & Y_\Gamma^* = \overline{Y}_\Gamma \\ & & = Y_\Gamma \amalg D \end{array}$ (univ. gen. elliptic curves (as grp. schs))	$\begin{array}{ccc} A & \hookrightarrow & \mathcal{G} \\ \downarrow & & \downarrow \uparrow e \\ Y_\Gamma & \hookrightarrow & \overline{Y}_\Gamma = Y_\Gamma \amalg D \end{array}$ (semi-abel.; smooth (non-can.) toroidal compact)
	$\underline{\omega} = e^* \Omega^1_{\mathcal{G}/\overline{Y}_\Gamma}$	$\underline{\omega} = e^* \Omega^1_{\mathcal{G}/\overline{Y}_\Gamma}$
	$S_k(\Gamma) = H^0(\overline{Y}_\Gamma, \underline{\omega}(-D))$	loc. free of rk. 2
		$S_k(\Gamma) = H^0(Y_{\overline{\Gamma}}, \underline{\omega}^k(-D))$

Over $\mathbb{Z}[\frac{1}{N}, \zeta_N] = R$ ← no longer for GSp_4

1) Mumford (GIT).

$N \geq 3$.

$$R\text{-Sch} \longrightarrow \text{Sets}$$

need loc. noe. → $$S \longmapsto \{(A, \lambda, \alpha)\}/\sim$$

$A \downarrow S$ ab. sch, λ prin. pol, $\alpha : (\mathbb{Z}/N\mathbb{Z})^4 \xrightarrow{\sim} A[N]$

std. sympl. pairing ${}^t x J y$; e_λ Weil pairing

is representable by $Y/_R$

quasi-proj., geom. connected, smooth, with univ. ab. $A \downarrow Y$

2) Faltings–Chai

$$\begin{array}{ccc} A & \hookrightarrow & \mathcal{G} \\ \downarrow & & \downarrow \\ Y & \hookrightarrow & \overline{Y} \\ & \searrow & \swarrow \pi \text{ proper} \\ & Y^* & \end{array}$$

$\mathcal{G}$ — semi-abel. sch.

$\overline{Y}$ — smooth toroidal comp. $/R$

$\overline{Y} = Y \amalg D$ — (rel.) divisor with normal crossing.

Y^* — normal, minimal compact.

$\overline{Y}$ needs the choice of an admiss. rational polyhedral cone decomp. of the cone $S_2(\mathbb{R})^+ \subset S_2(\mathbb{R})$ w.r.t. $\Gamma \subset Sp_4(\mathbb{Z})$. $U_Q = \begin{pmatrix} 1 & S_2(\mathbb{R}) \\ & 1 \end{pmatrix} \subset Sp_4(\mathbb{R})$

$Q = \begin{pmatrix} A & B \\ & {}^t A^{-1} \end{pmatrix} \supset$

$S \mapsto A S {}^t A$

$\overline{\Gamma} = \Gamma \cap Q / \Gamma \cap U_Q \subset GL_2(\mathbb{R})$

Mumford:

$$
\begin{tikzcd}
T = \underline{\mathrm{Isom}}(\mathbb{O}_Y^2, \underline{\omega}) \arrow[d, "GL_2"] \\
Y_{\Gamma/R}
\end{tikzcd}
\circlearrowleft GL_2 \qquad (\varphi, g) \longmapsto \varphi \circ g
$$