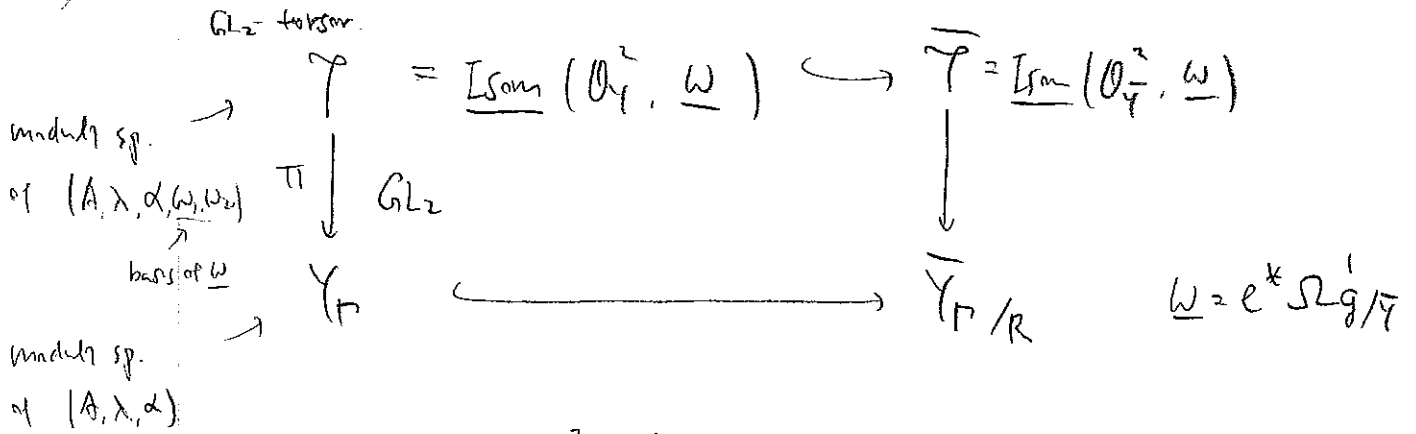


Tiloumel's lecture

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$$K = (k_1, k_2) \in \mathbb{Z}^2, \quad k_1 \geq k_2$$

$$\mathcal{N} \subset \mathcal{B} \subset \text{GL}_2 \subset \text{Sp}_4$$

$$\mathcal{T} \quad A \mapsto \left(\frac{A}{\det A} \right)$$

$$\frac{\omega^K}{h} = (\pi_x \circ \mathcal{O}_{\mathcal{T}})^{\vee}[-K]$$

$$\text{v.e.} \uparrow \quad (\text{or } (\pi_x \circ \mathcal{O}_{\overline{\mathcal{T}}})^{\vee}[-K] \text{ on } \overline{Y}_{\mathbb{F}})$$

$$\frac{\omega^K}{v} = \mathcal{T} \times^{\text{GL}_2} W_K \quad \rightarrow \text{need to choose an integral structure on } W_K$$

$$\uparrow$$

$$\text{v.e.} \quad (t \cdot g \cdot \omega) \sim (t \cdot \rho_k(g) \omega)$$

$$W_{K/\mathbb{R}} = \text{Ind}_{\mathcal{B}}^{\text{GL}_2} K = H^0(\text{GL}_2/\mathcal{B}, \mathcal{I}(K))$$

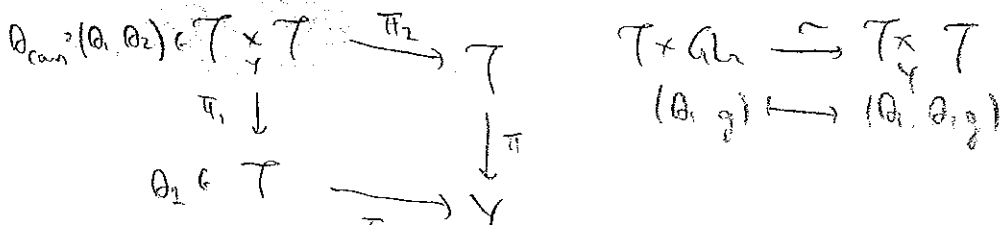
$$(\mathcal{I}(K) = \mathcal{O}_{\mathcal{B}} \times A^1(K))$$

$$= \{ f: \text{GL}_2/\mathcal{N} \rightarrow A^1_{\mathbb{R}}, f(gt) = K(t) f(g) \}$$

$$W_{K/\mathbb{C}} = \text{Sym}^{k_1-k_2} \otimes \det^{k_2}(\mathbb{C}^2)$$

If $p > k_1 + k_2$, then all these structures are the same / \mathbb{C}_p

Compare $\frac{\omega^K}{h}$ and W_v^K :



$(\pi_X \circ \theta_T)^* \omega$ consists of

$$f: T \times G_L \longrightarrow \mathcal{O}_T$$

$$h \in \mathcal{N} \quad f(\theta, gh) = f(\theta, g)$$

$$t \in T \quad f(\theta, gt) = (dt)^* f(\theta, g)$$

$$h \in G_L \quad f(\theta h, g) = f(\theta, hg)$$

\downarrow

$$f: G_L/\mu \longrightarrow \mathcal{O}_T$$

$$\text{s.t. } \rho_k(g)(f) = g^{-1} f \quad g \varphi(g') = \varphi(g^{-1} \cdot g')$$

\uparrow

$$\text{sections of } \begin{pmatrix} T^{G_L} \times \omega_k \\ \downarrow \\ Y \end{pmatrix} = \underline{\omega}_k$$

over \mathbb{C} .

$$\underline{\omega}_k = \Gamma \setminus (h_2 \times \omega_k(\mathbb{C}))$$

$$(\underline{\omega}_k)^{\text{an}} = \underline{\omega}_k$$

$$u^* \underline{\omega} \text{ trivial} \Rightarrow u^* T^{\text{an}} \cong h_2 \times G_L(\mathbb{C})$$

h_2

$$\Gamma \xrightarrow{u}$$

$\underline{\omega}$

Y_Γ

same code j(8.7)

$\underline{\omega}_k$ as a locally free sheaf over \overline{Y}_Γ

\overline{Y}

$$\underline{\omega}_k = \underline{\omega}_k(-D)$$

\downarrow

$$Y \subset \overline{Y} \supset D$$

$R \subset R'$, Arithmetic Siegel modular forms

R' = ring of def.
of Y_{Γ}

$$M_k(\Gamma, R') = H^0(Y_{\Gamma/R'}, \underline{\omega}^k) \\ = H^0(\overline{Y}_{\Gamma/R'}, \underline{\omega}^k)$$

Koecher's
principle

(For parallel weights, the global sections descend to the minimal compactification. For general weights, this is false)

$$S_k(\Gamma, R') = H^0(\overline{Y}_{\Gamma/R'}, \underline{\omega}^k) \text{ arithmetic Siegel cusp forms}$$

curves: $G_{L_2} = \mathfrak{h}_2 \hookrightarrow \mathbb{P}^2(\mathbb{C}) = M_{2,1}(\mathbb{C})^{(1)}/\mathbb{C}^*$

$$z \mapsto \begin{bmatrix} z \\ 1 \end{bmatrix}$$

$$\Omega_2(\mathbb{Q}) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mathbb{P}^1(\mathbb{Q})$$

$$\mathfrak{h}_2^* = \mathfrak{h}_2 \perp \mathbb{P}^1(\mathbb{Q})$$

$$Y_{\Gamma}^* = \overline{Y}_{\Gamma} = Y_{\Gamma} \perp \Gamma \backslash \mathbb{P}^1(\mathbb{Q})$$

$$\text{cusp of cusps} = \Gamma \backslash (\Omega_2(\mathbb{Q}) / \mathbb{R}/\mathbb{Q})$$

Grass: $\mathfrak{h}_2 \hookrightarrow \mathbb{L}(\mathbb{C}) = M_{4,2}(\mathbb{C}) / G_{L_2}(\mathbb{C})$

Lagrangian
planes of
 (\mathbb{C}^4, J)

$$\begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix} \begin{matrix} + \Omega_1 \Omega_2 \\ + \Omega_2 \Omega_1 \\ + \Omega_1 \Omega_2 \\ + \Omega_2 \Omega_1 \end{matrix} = 0$$

(Lagrangian condition)

$$\mathbb{L}(\mathbb{C}) = Sp_4(\mathbb{C}) / Q(\mathbb{C})$$

$Q =$ Siegel parabolic $\subset Sp_4$

$$Q = \text{Stab}_{Sp_4} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \left\{ \begin{pmatrix} A & 0 \\ 0 & A^{-t} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in Sp_4 \right\}$$

$$h_z^* = h_z \perp \text{Sp}_4(\mathbb{Q}) h_1 \perp \text{Sp}_4(\mathbb{Q}) h_0$$

$$\left(\begin{array}{c} z \\ 1 \end{array} \right) \quad \left(\begin{array}{c} 1 \\ z \\ 0 \\ 1 \end{array} \right) \quad \left(\begin{array}{c} 1 \\ 0 \end{array} \right)$$

$$(Y_\Gamma)^*_{\text{an}} = Y \perp Y_1 \perp Y_0$$

$\Gamma \backslash \text{Sp}_4(\mathbb{C}) / \text{Q}(\mathbb{Q})$
 = finite

3 dim \quad 1 dim \quad 0 dim
 finitely many open modular curves

q-exp. over \mathbb{C} at $\infty = \left[\frac{1}{0} \right]$.

Assume $\left(\begin{array}{c} 1 \\ s_2(z) \\ 1 \end{array} \right) \in \Gamma$

(analog of $\left(\begin{array}{c} 1 \\ 1 \end{array} \right) \in \Gamma$ for G_2)

$f \in M_k(\Gamma)$. $f(z+s) = f(z)$, $\forall s \in s_2(z)$

$$\begin{array}{ccc} s_2(z) & s_2(z)^* & \\ \cap & \cap & \\ s_2(\mathbb{R}) \times s_2(\mathbb{R}) & \longrightarrow & \mathbb{R} \\ (A, B) & \longrightarrow & \text{Tr}(AB) \end{array}$$

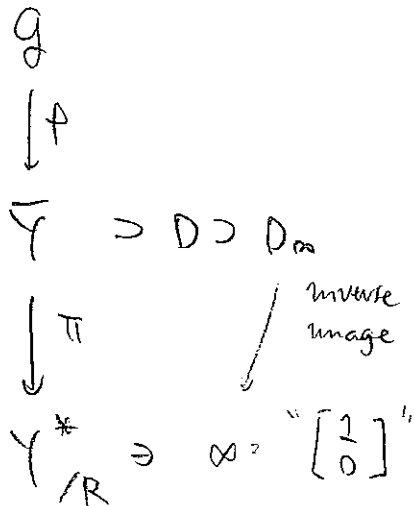
$$s_2^*(z) = \left\{ \left(\begin{array}{cc} a & b \\ b & c \end{array} \right) : \begin{array}{l} b \in \frac{1}{2}\mathbb{Z} \\ a, c \in \mathbb{Z} \end{array} \right\}$$

$$f(z) = \sum_{T \in s_2^*(z)} a_T e^{2\pi i \text{Tr}(TZ)}$$

Koecher: $a_T = 0$ unless $T \geq 0$.

f cuspidal $\Leftrightarrow a_T = 0$ unless $T > 0$

Arithmetic q-expansion



$\bar{Y}_{D_\infty}^{-1} = \bigcup_{\sigma \in \Sigma_\infty} U_\sigma$ affine open anal subscheme

$\mathbb{P} \subset Sp_4$ acts by $A \cdot S = A S A^{-1}$

$\bar{P} = \frac{\Gamma \cap Q}{\Gamma \cap U_Q} \subset \mathbb{A}_2(\mathbb{R})$ \bar{P} -admissible smooth real polyhedral cone

decomp of $S_2(\mathbb{R})^+ = \bigcup \bar{\sigma}$ (diagonal polyhedral)

$\bar{\sigma} \cap \bar{\sigma}' = \bar{\tau}$

\bar{P} acts on $\{\bar{\sigma}\}$.

$\{\bar{\sigma} \mid \bar{\sigma} \cap \bar{\sigma}' \neq \emptyset\}$ is finite

\bar{P}

in part. \bar{P} acts on Σ^∞

$U_\sigma \cong_{\bar{P}} S_\sigma = Sp_4 \mathbb{R} \left[\begin{array}{c} q^T \\ T \in S_2(\mathbb{Z})^* \cap \sigma^\vee \end{array} \right]$

smooth $\downarrow \forall S \in \sigma \subset S_2(\mathbb{R})^+, \text{tr}(S) \geq 0$

$\bigcap_{\sigma} (S_2(\mathbb{Z})^* \cap \sigma^\vee) = S_2(\mathbb{Z})^{*+}$

$$\phi_\sigma^* g \cong \mathbb{C}^2 / \underbrace{q_\sigma z^2}_{\prod_i q_{\sigma_i}^{q_i}}$$

$\mathbb{C} \rightarrow \mathbb{Y}$ "glued" from such things

$$\underline{\omega} \quad \frac{dq_{u,1}}{q_{u,1}} \quad \frac{dq_{u,2}}{q_{u,2}} \quad \text{basis for } \underline{\omega}$$

$$\begin{array}{ccc} f \in M_k(\Gamma, R') & \xrightarrow{\sim} & \phi_\sigma^* f \in H^0(S_\sigma, S_\sigma \otimes W_k) \\ \text{"} & & \\ H^0(\mathbb{Y}, \underline{\omega}^k) & & W_k \llbracket q^T; T \in S_\sigma(z)^* \cap \sigma^V \rrbracket \end{array}$$

$$\phi_\sigma^* f = f^* \in (W_k \llbracket q^T, T \in S_\sigma(z)^* \rrbracket)^{\overline{\Gamma}}$$

$$\left(\text{For } R = (0,0), \quad f^* \in \hat{\mathcal{O}}_{\mathbb{Y}, \infty}^* \right)$$

In p-adic setting, replace T by the Igusa tower.