Tilouine's lecture

$$\begin{array}{lcl}
\text{moduli sp. of } (A,\lambda,\alpha,\underline{\omega_1,\omega_2}) \text{ (basis of } \underline{\omega}) \to & \overset{GL_2\text{-torsor}}{\mathcal{T}} = \underline{\mathrm{Isom}}(\mathcal{O}_Y^2, \underline{\omega}) & \hookrightarrow \overline{\mathcal{T}} = \underline{\mathrm{Isom}}(\mathcal{O}_{\overline{Y}}^2, \underline{\omega}) \\
 & \pi \downarrow GL_2 & \downarrow \\
\text{moduli sp. of } (A,\lambda,\alpha) \to & Y_\Gamma & \longrightarrow \overline{Y}_{\Gamma/R} \qquad \underline{\omega} = e^* \Omega^1_{g/\overline{Y}}
\end{array}$$

$K = (k_1, k_2) \in \mathbb{Z}^2$, $k_1 \geq k_2$ $\qquad \underbrace{N \subset B}_{T} \subset GL_2 \subset Sp_4 \qquad A \longmapsto \begin{pmatrix} A & \\ & {}^tA^{-1} \end{pmatrix}$

$\underline{\omega}^K_h = (\pi_* \mathcal{O}_{\mathcal{T}})^N[-K]$

h.l.

$\left(\text{or } (\pi_* \mathcal{O}_{\overline{\mathcal{T}}})^N[-K] \text{ on } \overline{Y}_\Gamma\right)$

$\underline{\omega}^K_v = \mathcal{T} \overset{GL_2}{\times} W_K$ $\to$ need to choose an integral structure on this W_K

v.b. $\qquad (t \cdot g, w) \sim (t, \rho_K(g) w)$

$W_{K/R} = \mathrm{Ind}_B^{GL_2} K = H^0(GL_2/B, \mathcal{L}(K))$

$\qquad (\mathcal{L}(K) : GL_2 \overset{B}{\times} \mathbb{A}^1(K))$

$= \{ f : GL_2/N \to \mathbb{A}^1_R,\ f(gt) = K(t) f(g) \}$

$W_{K/\mathbb{C}} = \mathrm{Sym}^{k_1-k_2} \otimes \det^{k_2} (\mathbb{C}^2)$

If $p > k_1+k_2$, then all these structures are the same $/\mathbb{C}_p$

Compare $\underline{\omega}^K_h$ and $\underline{\omega}^K_v$:

$$\begin{array}{ccc}
\theta_{can} : (\theta_1, \theta_2) \in \mathcal{T} \times_Y \mathcal{T} & \xrightarrow{\pi_2} & \mathcal{T} \\
\pi_1 \downarrow & & \downarrow \pi \\
\theta_1 \in \mathcal{T} & \xrightarrow{\pi} & Y
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{T} \times GL_2 & \xrightarrow{\sim} & \mathcal{T} \times_Y \mathcal{T} \\
(\theta, g) & \longmapsto & (\theta, \theta g)
\end{array}$$

$(\pi_* \mathcal{O}_{\mathcal{T}})^{N}[-k]$ consists of

$$f : \mathcal{T} \times GL_2 \longrightarrow \mathcal{O}_Y$$

$n \in N \quad f(\theta, gn) = f(\theta, g)$

$t \in T \quad f(\theta, gt) = k(t)^{-1} f(\theta, g)$

$h \in GL_2 \quad f(\theta h, g) = f(\theta, hg)$

$$\updownarrow$$

$$\underline{f} : GL_2/N \longrightarrow \mathcal{O}_{\mathcal{T}}$$

s.t. $\rho_k(g)(\underline{f}) = g^{-1}\underline{f} \qquad g\varphi(g') = \varphi(g^{-1} \cdot g')$

$$\updownarrow$$

Sections of $\begin{pmatrix} \mathcal{T}^{GL_2} \times W_k \\ \downarrow \\ Y \end{pmatrix} = \underline{\omega}_v^{k}$

over $\mathbb{C}$

$$\underline{\omega}^{k} = \Gamma \backslash (\mathfrak{h}_2 \times W_k(\mathbb{C}))$$

$$(\underline{\omega}_v^{k})^{an} = \underline{\omega}^{k}$$

$u^* \underline{\omega}$ trivial $\Rightarrow u^* \mathcal{T}^{an} \cong \mathfrak{h}_2 \times GL_2(\mathbb{C})$

$$\mathfrak{h}_2 \xrightarrow{u} Y_\Gamma \qquad \Gamma \qquad \underline{\omega}$$

same cocycle $j(\gamma, \tau)$

$\underline{\omega}^k$ as a locally free sheaf over $\overline{Y}_\Gamma$

$$\overline{\mathcal{T}} \qquad \underline{\omega}_k = \underline{\omega}^k(-D)$$

$$\downarrow$$

$$Y \subset \overline{Y} \supset D$$

$R \subset R'$. Arithmetic Siegel modular forms

R' = ring of def. of Y_Γ

$$M_k(\Gamma, R') = H^0(Y_{\Gamma/R'}, \underline{\omega}^k)$$
$$= H^0(\overline{Y}_{\Gamma/R'}, \underline{\omega}^k)$$

Koecher's principle

(For parallel weight, the global sections descend to the minimal compactification. For general weight, this is <u>false</u>)

$S_k(\Gamma, R') = H^0(\overline{Y}_{\Gamma/R'}, \underline{\omega}_k)$ arithmetic Siegel cusp forms

<u>Cusps</u>: <u>GL_2</u>: $\mathfrak{h}_1 \hookrightarrow \mathbb{P}^1(\mathbb{C}) = M_{2,1}(\mathbb{C})^{(1)}/\mathbb{C}^\times$

$$z \longmapsto \begin{bmatrix} z \\ 1 \end{bmatrix}$$

$$SL_2(\mathbb{Q}) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mathbb{P}^1(\mathbb{Q}).$$

$$\mathfrak{h}_2^* = \mathfrak{h}_1 \amalg \mathbb{P}^1(\mathbb{Q})$$

$$Y_\Gamma^* = \overline{Y}_\Gamma = Y_\Gamma \amalg \Gamma \backslash \mathbb{P}^1(\mathbb{Q})$$

cusp of cusps $= \Gamma \backslash SL_2(\mathbb{Q})/B(\mathbb{Q})$

<u>GSp_4</u>. $\mathfrak{h}_2 \hookrightarrow \mathcal{L}(\mathbb{C}) = M_{4,2}^{(2) \to \text{Lagrangian}}(\mathbb{C}) / GL_2(\mathbb{C})$

Lagrangian planes of $(\mathbb{C}^4, J)$

$\begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix}$ ${}^t\Omega_1\Omega_2 = {}^t\Omega_2\Omega_1$, ${}^t\Omega J \Omega = 0$. (Lagrangian condition)

$$\mathcal{L}(\mathbb{C}) = Sp_4(\mathbb{C})/Q(\mathbb{C})$$

Q = Siegel parabolic $\subset Sp_4$

$$Q = \mathrm{Stab}_{Sp_4}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \left\{ \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix}\begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} \subset Sp_4 \right\}$$

S sym

$$\mathfrak{h}_2^* = \mathfrak{h}_2 \amalg Sp_4(\mathbb{Q})\,\mathfrak{h}_1 \amalg Sp_4(\mathbb{Q})\,\mathfrak{h}_0$$

$$\mathfrak{h}_2 = \left\{\begin{bmatrix} Z \\ 1 \end{bmatrix}\right\} \qquad \mathfrak{h}_1 = \left\{\begin{bmatrix} 1 & & & \\ & z & & \\ 0 & & & \\ & & & 1 \end{bmatrix}\right\} \qquad \mathfrak{h}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$(Y_\Gamma^*)_{an} = Y \amalg Y_1 \amalg Y_0$$

Y_1: finitely many open modular curves

$Y_0 = \Gamma \backslash Sp_4(\mathbb{C})/Q(\mathbb{Q})$ finite

3 dim'l 1 dim'l 0 dim'l

q-exp. over $\mathbb{C}$ at $\infty = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Assume $\begin{pmatrix} 1 & S_2(\mathbb{Z}) \\ & 1 \end{pmatrix} \subset \Gamma$

(analog of $\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \in \Gamma$ for GL_2)

$f \in M_k(\Gamma)$. $f(Z+S) = f(Z)$, $\forall S \in S_2(\mathbb{Z})$

$$\begin{array}{ccccc} S_2(\mathbb{Z}) & & S_2(\mathbb{Z})^* & & \\ \cap & & \cap & & \\ S_2(\mathbb{R}) & \times & S_2(\mathbb{R}) & \longrightarrow & \mathbb{R} \\ (A & , & B) & \longmapsto & Tr(AB) \end{array}$$

$$S_2^*(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} : \begin{array}{l} b \in \frac{1}{2}\mathbb{Z} \\ a, c \in \mathbb{Z} \end{array} \right\}$$

$$f(Z) = \sum_{T \in S_2(\mathbb{Z})^*} a_T \, e^{2\pi i \, Tr(TZ)}$$

Koecher: $a_T = 0$ unless $T \geq 0$.

f cuspidal $\iff a_T = 0$ unless $T > 0$

Arithmetic q-expansion

$$\begin{array}{l} \mathcal{G} \\ \big\downarrow \phi \\ \bar{Y} \supset D \supset D_\infty \\ \big\downarrow \pi \qquad\qquad \text{inverse image} \\ Y^*_{/R} \ni \infty = \text{"}\begin{bmatrix}1\\0\end{bmatrix}\text{"} \end{array}$$

$$\hat{\bar{Y}}_{D_\infty} = \bigcup_{\sigma \in \Sigma_\infty} U_\sigma \quad \text{affine open formal subscheme}$$

$\Gamma \subset Sp_4$ — acts by $A \cdot S = A S {}^tA$

$\bar{\Gamma} = \dfrac{\Gamma \cap Q}{\Gamma \cap U_Q} \subset GL_2(\mathbb{R})$ — $\bar{\Gamma}$-admissible smooth rat'l polyhedral cone decomp of $S_2(\mathbb{R})^+ = \bigcup \bar{\sigma}$

disjoint ← polyhedral

$\bar{\sigma} \cap \bar{\sigma}' = \bar{\tau}$

$\bar{\Gamma}$ acts on $\{\sigma\}$,

$\{\gamma \mid \sigma \cap \gamma\sigma \neq \phi\}$ is finite
$\gamma \in \bar{\Gamma}$

in part. $\bar{\Gamma}$ acts on Σ^∞

$$\mathcal{U}_\sigma \overset{p_\sigma}{\cong} \mathcal{S}_\sigma := \mathrm{Spf}\, R[[q^T ;\ T \in S_2(\mathbb{Z})^* \cap \sigma^\vee]]$$

$\forall S \in \sigma \subset S_2(\mathbb{R})^+,\ \mathrm{Tr}(ST) \geq 0$

smooth

$$\bigcap_\sigma (S_2(\mathbb{Z})^* \cap \sigma^\vee) = S_2(\mathbb{Z})^{*+}$$

$$\varphi_\sigma^* \mathcal{G} \cong \mathbb{G}_m^2 / \underline{q_\sigma}^{\mathbb{Z}^2} \quad \prod_i q_{\eta_i}^{a_i}$$

$\mathcal{G} \downarrow \overline{Y}$ "glued" from such things

$\underline{\omega}$ $\quad \frac{dq_{v,1}}{q_{v,1}}, \frac{dq_{v,2}}{q_{v,2}}$ basis for $\underline{\omega}$

$$f \in M_k(\Gamma, R') = H^0(\overline{Y}, \underline{\omega}^k) \rightsquigarrow \varphi_\sigma^* f \in H^0(S_\sigma, S_\sigma \otimes W_k) \quad W_k[\![q^T ; T \in S_2(\mathbb{Z})^* \cap \sigma^\vee]\!]$$

$$\varphi_\sigma^* f = f^* \in \left(W_k [\![q^T, T \in S_2(\mathbb{Z})^{*,+}]\!]\right)^\Gamma$$

(For $k = (a, a)$, $f^* \in \hat{\mathcal{O}}_{Y^*, \infty}$)

In p-adic setting, replace T by the Igusa tower.