Hida's theory for GL(2) and GSp(4)

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1. Introduction

Let $p \geq 5$ be a prime, and N be an integer prime to p, called the tame level. For each weight $k \geq 2$, and integer $r \geq 1$, consider the spaces of cusp forms of level Np^r with $\mathbf{Z}_p$ coefficients:

$$S_k(\Gamma_1(Np^r), \mathbf{Z}_p), r \geq 1$$

For each pair of integers $r \geq s \geq 1$, we have the inclusion map:

$$S_k(\Gamma_1(Np^s), \mathbf{Z}_p) \to S_k(\Gamma_1(Np^r), \mathbf{Z}_p)$$

The basic question is: how can we organise these spaces with respect to these inclusions?

Here, in the case of Hida's theory, we'll limit to the space of *ordinary* forms: let $S_k^{\mathrm{ord}}(\Gamma_1(Np^r), \mathbf{Z}_p)$ be the largest $\mathbf{Z}_p$ submodule of $S_k(\Gamma_1(Np^r), \mathbf{Z}_p)$ over which the operator U_p acts as isomorphism. As above, we have the inclusion:

$$S_k^{\mathrm{ord}}(\Gamma_1(Np^s), \mathbf{Z}_p) \to S_k^{\mathrm{ord}}(\Gamma_1(Np^r), \mathbf{Z}_p), r \geq s \geq 1 \tag{1.1}$$

Let $S_k^{\mathrm{ord}}(\Gamma_1(Np^\infty), \mathbf{Z}_p) := \bigcup_{r=1}^{\infty} S_k^{\mathrm{ord}}(\Gamma_1(Np^r), \mathbf{Z}_p)$. Define $\bar{S}_k^{\mathrm{ord}}(\Gamma_1(Np^\infty), \mathbf{Z}_p)$, the ring of ordinary p-adic modular forms, to be the completion of $S_k^{\mathrm{ord}}(\Gamma_1(Np^\infty), \mathbf{Z}_p)$ with respect to the sup norm on Fourier coefficients. We would like to know the structure of $\bar{S}_k^{\mathrm{ord}}(\Gamma_1(Np^\infty), \mathbf{Z}_p)$ and the relation between $\bar{S}_k^{\mathrm{ord}}(\Gamma_1(Np^\infty), \mathbf{Z}_p)$ and $S_k^{\mathrm{ord}}(\Gamma_1(Np^r), \mathbf{Z}_p)$.

Hida studied these spaces through their Hecke algebras. Let $h_k(Np^r, \mathbf{Z}_p)$ be the Hecke algebra of level Np^r, which is a semilocal $\mathbf{Z}_p$-algebra, acting on $S_k(\Gamma_1(Np^r), \mathbf{Z}_p)$. Define $h_k^{\mathrm{ord}}(Np^r, \mathbf{Z}_p)$, the ordinary part, to be the maximal direct summand over which the image of U_p is invertible.

These Hecke algebras are dual to the space of cusp forms via the perfect pairing:

$$\langle , \rangle : h_k^{\mathrm{ord}}(Np^r, \mathbf{Z}_p) \times S_k^{\mathrm{ord}}(\Gamma_1(Np^r), \mathbf{Z}_p) \to \mathbf{Z}_p$$

given by $\langle h, f \rangle = a(1, f|h)$.

Dual to the inclusion (1.1) are the restriction maps:

$$h_k^{\mathrm{ord}}(\Gamma_1(Np^r), \mathbf{Z}_p) \to h_k^{\mathrm{ord}}(\Gamma_1(Np^s), \mathbf{Z}_p), r \geq s \geq 1 \tag{1.2}$$

And the above pairing is compatible with the transition maps (1.1) and (1.2).

Thus, the structure of $\bar{S}_k^{\mathrm{ord}}(\Gamma_1(Np^\infty), \mathbf{Z}_p)$ can be obtained, by duality, from the structure of:

$$h_k^{\mathrm{ord}}(\Gamma_1(Np^\infty), \mathbf{Z}_p) = \lim_{\leftarrow} h_k^{\mathrm{ord}}(\Gamma_1(Np^r), \mathbf{Z}_p)$$

Recall the group $\Gamma = 1+p\mathbf{Z}_p$, let $\Gamma_r = 1+p^r\mathbf{Z}_p$ be the unique subgroup of index p^{r-1}, so that Γ/Γ_r is identified as the p-primary component of $(\mathbf{Z}/p^r\mathbf{Z})^*$. Consider the Iwasawa algebra $\Lambda = \mathbf{Z}_p[[\Gamma]] := \lim_{\leftarrow} \mathbf{Z}_p[\Gamma/\Gamma_r]$ The space $\bar{S}_k^{\mathrm{ord}}(\Gamma_1(Np^\infty), \mathbf{Z}_p)$ can be given the structure of Λ module, using the fact that, at a finite level, the diamond operators gives $S_k^{\mathrm{ord}}(\Gamma_1(Np^r), \mathbf{Z}_p)$ the structure of $\mathbf{Z}_p[\Gamma/\Gamma_r]$ module. Dual to this, the algebra $h_k^{\mathrm{ord}}(\Gamma_1(Np^\infty), \mathbf{Z}_p)$ can be given the strucure of Λ algebra as follows: we have a (ring) homomorphism:

$$\mathbf{Z}_p[\Gamma/\Gamma_r] \rightarrow h_k^{\mathrm{ord}}(\Gamma_1(Np^r), \mathbf{Z}_p)$$

given by the diamond operators. Passing to the inverse limit, we get a homomorphism:

$$\Lambda = \lim_{\leftarrow} \mathbf{Z}_p[\Gamma/\Gamma_r] \rightarrow h_k^{\mathrm{ord}}(\Gamma_1(Np^\infty), \mathbf{Z}_p)$$

This gives the desired Λ algebra structure on Λ. The following theorem of Hida gives us information on the structure of $h_k^{\mathrm{ord}}(\Gamma_1(Np^r), \mathbf{Z}_p)$, via that of $h_k^{\mathrm{ord}}(\Gamma_1(Np^\infty), \mathbf{Z}_p)$:

Theorem 1.1. *The Λ module $h_k^{\mathrm{ord}}(\Gamma_1(Np^\infty), \mathbf{Z}_p)$ is free over Λ of finite rank; moreover the perfect control theorem holds, i.e. for all $r \geq 1$, we have:*

$$h_k^{\mathrm{ord}}(\Gamma_1(Np^r), \mathbf{Z}_p) \cong h_k^{\mathrm{ord}}(\Gamma_1(Np^\infty), \mathbf{Z}_p) \otimes_\Lambda \mathbf{Z}_p[\Gamma/\Gamma_r]$$

Note that, by duality, this theorem gives a corresponding result on $\bar{S}_k^{\mathrm{ord}}(\Gamma_1(Np^\infty), \mathbf{Z}_p)$, that it's a free Λ module of finite rank, and $S_k^{\mathrm{ord}}(\Gamma_1(Np^r), \mathbf{Z}_p)$ is precisely its submodule of Γ_r invariants.

Furthermore, the fact that $h_k^{\mathrm{ord}}(\Gamma_1(Np^\infty), \mathbf{Z}_p)$ is finite free over Λ means that $\mathrm{Spec}(h_k^{\mathrm{ord}}(\Gamma_1(Np^\infty), \mathbf{Z}_p))$ is a finite (ramified) cover of $\mathrm{Spec}(\Lambda)$, the weight space. Since a p-ordinary eigenform is just a $\overline{\mathbf{Z}}_p$ valued point of $\mathrm{Spec}(h_k^{\mathrm{ord}}(\Gamma_1(Np^\infty), \mathbf{Z}_p))$, one can deduce that an eigenform can always be p-adically deformed in families.

Two approaches, one algebro-geometric and other being group cohomological, were given by Hida to prove Theorem 1.1. In the group cohomological approach [**2**], one studies the cohomology of the groups $\Gamma_1(Np^r)$ with coefficients in certain $\mathbf{Z}_p$ modules. The Hecke algebra $h_k^{\mathrm{ord}}(\Gamma_1(Np^\infty), \mathbf{Z}_p)$ acts naturally on these cohomology groups. One establish the control theorems for these cohomology groups, and then deduce the control theorems for the Hecke algebras. This approach has been generalised in [**5**] to the case of GSp(4). However, the existence of p-torsion in the higher cohomology groups in the more general case presents some difficulties [**4**].

The algebro-geometric one [**1**], which is historically the first, one directly establishes the control theorem for the spaces of cusp forms. For this, one need to study the Igusa tower, which form a covering of the ordinary locus of modular curves. This approach has been generalised in [**3**] [**4**] to quite general Shimura Varieties.

References

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