Chenevier's lectures on eigenvarieties of definite unitary groups

Chung Pang Mok

1. Introduction

There will be three objectives in Chenevier's lectures:

(1) construction of the eigenvarieties of definite unitary groups of any rank.
(2) study of the local and global properties of the families of Galois representations on the eigenvarieties.
(3) applications to standard conjectures relating arithmetic L-functions at integers and associated Selmer groups.

Part (3) (lectures 6, 7) would be the main motivation of the lectures. A basic conjecture in this area is the Bloch-Kato conjecture. If V is a geometric Galois representation, then it predicts a connection between the complex analytic L function $L(s,V)$ of V, and an object of arithmetic nature, the Selmer group $H^1_f(V^*(1))$:

$$\mathrm{ord}_{s=0}L(s,V) = \dim H^1_f(V^*(1)) - \dim H^0(V^*(1))$$

In the lectures concerning (3), we will first explain a method to construct a non trivial element in $H^1_f(V)$ when V is pure, satisfies $V^*(1) = V$, and when the L-function vanishes for a sign reason at the center of its functional equation. We will moreover focus on the problem of constructing linearly independent elements in $H^1_f(V)$. Our main result here will be that if an explicit eigenvariety is non smooth at some explicit point, then $H^1_f(V)$ has rank ≥ 2. This work is based on a detailled study of the family of Galois representations on definite unitary eigenvarieties round some reducible points, which will be done in part (1) and (2). The techniques of using p-adic families (or congruences) to construct "interesting" arithmetic elements was first employed by Ribet [**7**] and Wiles [**8**] in the work on Iwasawa main conjecture.

Part (1) (lectures 3 ,4, 5) is an extension of works of Hida (the ordinary case) and Coleman (general finite slope case, but in rank 2). Using ideas of Coleman [**4**] and of Buzzard [**2**] , we will explain a construction of p-adic families of finite slope p-adic eigenforms, and the eigenvarieties, on any definite unitary groups.

Part (2) is fundamental to link the information on eigenvarieties to that of Galois cohomology. We'll explain for example how the Bloch-Kato conjectures enter into the picture, and what they say at the non critical classical points of the eigenvarieties. In this part, two difficulties arise.

The author is grateful to Chenevier for most of the details.

The first one is that (for some good reasons) the families of Galois representations on the eigenvarieties do not exist as true representations, but only in the weak sense of traces (called *pseudo-characters*, first employed by Wiles in the context of Hida's theory for Hilbert modular forms). For example, this causes some difficulties in generalising Ribet's technique. An important ingredient here is a general structure theorem for residually multiplicity free pseudocharacters with values in any henselian local ring, allowing to describe their reducibility loci and the associated Ext-groups (see chapter 1 of [**1**]).

The second one (lectures 1, 2) is that the local properties at p of the Galois families. Over the ordinary locus of the eigenvarieties, the representation is ordinary at p. On the other hand, the situation is much more subtle over the non-ordinary locus, and still partly unknown. In recent years, the works of Kisin [**6**] on the Fontaine-Mazur conjecture, and of Colmez [**5**] on trianguline representations, give important insight to this question. We will explain (see chapter 2 of [**1**]) how these works extend to the higher dimensional case.

References

[1] J.Bellaïche, G. Chenevier, *p-adic families of Galois Representations and higher rank Selmer groups*, preprint.

[2] K.Buzzard, *Eigenvarieties*, preprint.

[3] G.Chenevier, *Families p-adiques de formes automorphes pour* $\mathrm{GL}(n)$, Journal für die reine und angewandte Mathematik, **570** (2004), 143-217.

[4] R.Coleman, *p-adic Banach spaces and families of modular forms*, Invent.Math. **124** (1996), 214-241.

[5] P.Colmez, *Séries principale unitaire pour* $\mathrm{GL}_2(\mathbf{Q}_p)$ *et représentations triangulines de dimension 2*, preprint.

[6] M.Kisin, *Overconvergent modular forms and the Fontaine-Mazur conjecture*, Invent. Math. **153** 2003, 373-454.

[7] K.Ribet, *a modular construction of unramified extension of* $\mathbf{Q}(\mu_p)$ Invent.Math. **34**, 1976, no.3, 151-162.

[8] A.Wiles, *The Iwasawa conjecture for totally real fields*493-540. Annals of Mathematics, **131**, 1990, no.3,

Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA 02138

E-mail address: mok@fas.harvard.edu