Schneider's lectures on p-adic Banach representations

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1. Introduction

Let K be a finite field extension of $\mathbf{Q}_p$, with residue field k. Local class field theory seeks to understand the abelian extensions of K, i.e. the structure of the group $\mathrm{Gal}(\bar{K}/K)^{ab}$, in terms of data from the ground field K. Recall the main theorem of local class field theory: there is a continuous injective homomorphism, called the local reciprocity map:

$$r_K : K^\times \to \mathrm{Gal}(\bar{K}/K)^{ab}$$

which takes uniformizers of K to Frobenius elements (i.e. elements of $\mathrm{Gal}(\bar{K}/K)$ which act as the Frobenius $x \to x^{|k|}$ on $\bar{k}$). To characterise the image, recall the Weil group, W_K, which is defined as the subgroup of $\mathrm{Gal}(\bar{K}/K)$ whose elements act as integer powers of the Frobenius on $\bar{k}$. One can define a topology on W_K, in such a way that the local reciprocity map induces a topological isomorphism:

$$r_K : K^\times \xrightarrow{\sim} W_K^{ab}$$

In the sixties, Langlands proposed to generalise class field theory beyond the abelian extensions, by taking the dual point of view: in the example of local class field theory above, one can view the reciprocity map as giving a correspondence between continuous complex valued characters of W_K (which necessarily factors through W_K^{ab}) and continuous complex valued characters of $K^\times = \mathrm{GL}_1(K)$.

Langlands then proposed, in general, that given an irreducible continuous representation of W_K on n-dimensional complex vector space, there should correspond a certain type of representation of the p-adic Lie group $\mathrm{GL}_n(K)$ on complex vector space, called irreducible smooth representations, such that, under this correspondence, the invariants constructed from both sides should match. This is known as the Local Langlands Conjecture.

The Local Langlands Conjecture was proved by Henniart, and Harris-Taylor a few years ago. However, this apparent climax in fact is, in the light of new developments, far from being the end of the story. As an illustration, one sees that under any finite dimensional complex representation of W_K, the image of the wild inertia, a pro-p subgroup of the inertia group, is always a finite group, essentially by topological reasons. We would get a better match of topologies if we consider, instead, *continuous* representations of W_K on n-dimensional p-adic vector spaces. But what kind of representations of $\mathrm{GL}_n(K)$ should they correspond to?

Motivated by this (and by other number theoretical problems), there's an increasing interest in recent years in developing the representation theory of p-adic Lie groups on locally convex vector spaces over p-adic fields. The first task at hand is to construct a reasonable category of such representations, which includes all the examples one is interested in, but which still is manageable. As very often in representation theory one interprets representations as modules over some kind of group algebra. The idea then is to impose module theoretic finiteness conditions. In our case these algebras are p-adic distributions algebras. There are a number of very basic difficulties one has to deal with. For example, these algebras are very big. Even worse, p-adic Lie groups do not carry p-adic valued Haar measures. This means most of the traditional methods in the harmonic analysis on real Lie groups cannot be imitated. In the work Schneider-Teitelbaum they seem to have constructed the right category.

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