Kisin's lectures on the eigencurve via Galois representations

Chung Pang Mok

1. Introduction

The aim of the series of lectures is to explain progress towards the Fontaine-Mazur conjecture for GL_2 and how it ties in with a Galois representation construction of the eigencurve. The general plan is as follows:

1: Overview of main results.
2: The eigencurve via Galois representations.
3: Classification of crystalline Galois representations. (2 lectures).
4: Construction of potentially semi-stable deformation ring (2 lectures).
5: Modularity of potentially Barsotti-Tate representations.
6: The Fontaine-Mazur conjecture for GL_2 (2 lectures).

Recall the statement of the Fontaine-Mazur conjecture, which says that a continuous two dimensional $\mathbb{Q}_p$-representation ρ of $G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, which is odd irreducible, unramified outside finitely many primes, and whose restriction to the decomposition group at p is potentially semi-stable, necessarily arises from a modular form.

A good test case for the conjecture is when ρ comes from an overconvergent eigenform f of finite slope and integral weight $k \geq 2$. Such representations do not come from true modular forms, but they can be closely approximated by representations which do. The main result of [**5**] is that, apart from a certain exceptional case, the Fontaine-Mazur conjecture is true for these representations. To prove this one shows that any finite slope, overconvergent eigenform f admits a crystalline period with Frobenius acting by the U_p eigenvalue λ. This is a kind of p-adic interpolation of a result of Saito [**10**]. When the representation attached to f is potentially semi-stable, this implies that $\mathrm{val}_p(\lambda) \leq k-1$, and a result of Coleman [**3**] guarantees that f is classical, other than in a certain exceptional case. The condition that a two dimensional Galois representation has such a period can be used to define a rigid analytic space X_{fs} which is expected to coincide with the eigencurve.

The remaining lectures will discuss progress towards the Fontaine-Mazur conjecture. There are two approaches, one which works for potentially Barsotti-Tate representations, and one which should work for essentially all potentially semi-stable representations, but which currently requires (among other things) that the extension over which the representation becomes semi-stable.is abelian. Both approaches rely on a classification of semi-stable Galois representations over ramified extensions of $\mathbb{Q}_p$. We will explain this classification which uses ideas of Berger and of Breuil, and relies on Kedlaya's theory of slopes [**7**]

The first approach to modularity involves studying flat deformation rings by building over them a kind of resolution which parameterizes finite flat group schemes [**6**]. This uses the above classification of crystalline representations directly, and works well for potentially Barsotti-Tate representations. We will explain this as well as the modification of the Taylor-Wiles method when the local deformation ring is no longer a power series.

For more general potentially semi-stable representations one cannot use the classification directly, however one can use it to construct the relevant local deformation rings [**8**]. Modularity then becomes equivalent to a conjecture of Breuil-Mézard [**2**] on the Hilbert-Samuel multiplicity of the mod p reduction of this ring. The proof of the Breuil-Mézard conjecture [**9**] relies on recent developments in the p-adic local Langlands correspondence due to Colmez and Berger-Breuil [**4**], [**1**]. The results of this second method are strong enough to imply that X_{fs} is equal to the eigencurve in many cases.

References

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Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA 02138

E-mail address: mok@math.harvard.edu