Taylor's lecture on proving modularity without Ihara's lemma

Chung Pang Mok

1. Introduction

Taylor will give three lectures, to be held respectively on April 5 at 1:00, April 7 at 3:00, and on April 14 at 3:00. The title is "Proving modularity without Ihara's lemma", which is the subject of the preprint [**3**]. Furthermore, Harris and Taylor will give lectures on April 21, 2:00 to 4:00, entitled "A family of Calabi-Yau varieties and the Sato-Tate conjecture".

The aim of Taylor's lectures is to generalize the modularity lifting theorems of Wiles [**5**] and Taylor-Wiles [**4**] to higher dimensional Galois representations.

Recall the works of [**4**], [**5**]. Let

$$\bar{\rho} : \mathrm{Gal}(\bar{\mathbb{Q}}, \mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F}_l)$$

be a irreducible, 2-dimensional mod l Galois representation, unramified outside a finite set of primes S. Assume that it is modular.

Let

$$\rho : \mathrm{Gal}(\bar{\mathbb{Q}}, \mathbb{Q}) \to \mathrm{GL}_2(\mathbb{Z}_l)$$

be an l-adic lifting of $\bar{\rho}$ (i.e. ρ mod $l \cong \bar{\rho}$). One would like to prove that ρ is also modular. Here, the set of primes at which ρ ramifies may be larger than that of $\bar{\rho}$.

Wiles approached the problem through deformation theory: for each finite set of primes Σ (which can be empty), there's a universal deformation ring R_Σ which parametrises the liftings of $\bar{\rho}$, with prescribed ramifications outside the set Σ (these representations are, in particular, unramified outside $S \cup \Sigma$). On the other hand, one constructs a Hecke ring $\mathbb{T}_\Sigma$, which parametrises those which are modular. One would like to prove an isomorphism $R_\Sigma \cong \mathbb{T}_\Sigma$, which is a precise way of saying that all such liftings of $\bar{\rho}$ are modular.

Wiles discovered a numerical criterion for the two rings to be isomorphic: he constructed numerical invariants for both R_Σ, the Galois side, and $\mathbb{T}_\Sigma$, the Hecke side. In [**5**] it is proved that these two invariants are equal iff $R_\Sigma \cong \mathbb{T}_\Sigma$.

In [**4**] (especially the appendix), Taylor and Wiles proved the isomorphism $R_\emptyset \cong \mathbb{T}_\emptyset$ directly (i.e. without using the numerical criterion), hence the two numerical invariants are equal in the case $\Sigma = \emptyset$, called the minimal case. Their methods depend essentially on the assumption $\Sigma = \emptyset$.

By studying how the two numerical invariants change when one pass from a set of primes Σ to a larger set Σ', one can show that the equality of the invariants in the case of Σ implies equality in the case of Σ'. This part of the argument is known as "level raising". Here, important use is made of the Ihara's lemma, which gives the relation between the Hecke side invariants for Σ and Σ'.

In the preprint [**1**] (but note that the results of this paper are known for many years) , the aim of the authors is to generalize the modularity lifting arguments to certain n-dimensional mod l Galois representations, which arise from cuspidal automorphic representation of GL_n. They formulated a conjectural generalization of Ihara's lemma, which, if true, would allow most of the level raising arguments of Wiles to carry over (they were able to generalize the arguments of Taylor-Wiles unconditionally).

The next important development came with the work of Kisin [**2**]. He considered framed deformation problems, and that one could hope to prove modularity if the local deformation ring was only integral but not smooth (via modification of the Taylor-Wiles method when the local deformation ring is no longer a power series).

In the preprint [**3**], Taylor generalizes the methods of Kisin to the higher dimension case, and proves the main results of [**1**] unconditionally.

The importance of these results cannot be overestimated. As one of the highlights, we now have a proof of the Sato-Tate conjecture for elliptic curves over totally real fields which are semistable at some prime, a conjecture which is not known even for a single example before. The deduction of this from the modularity lifting results will be the subject of the lectures by Harris and Taylor.

References

[1] L.Clozel, M.Harris, and R.Taylor, *Automorphy for some l-adic lifts of automorphic mod l Galois representations*, preprint. (2005)

[2] M.Kisin, *Moduli of finite flat group schemes, and modularity*, preprint. (2004)

[3] R.Taylor, *Automorphy for some l-adic lifts of automorphic mod l Galois representations II*, preprint. (2006)

[4] R.Taylor and A.Wiles, *Ring theoretic properties of certain Hecke algebras*, Ann. of Math. 141(1995).

[5] A.Wiles, *Modular elliptic curves and Fermat's Last Theorem*, Ann. of Math. 141(1995).

Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA 02138

E-mail address: mok@math.harvard.edu