QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Tuesday September 18 2007 (Day 1)

1. Let $f(x) = x^4 - 7 \in \mathbb{Q}[x]$.

(a) Show that f is irreducible in $\mathbb{Q}[x]$.

(b) Let K be the splitting field of f over $\mathbb{Q}$. Find the Galois group of $K/\mathbb{Q}$.

(c) How many subfields $L \subset K$ have degree 4 over $\mathbb{Q}$? How many of them are Galois over $\mathbb{Q}$?

2. A real-valued function f defined on an interval $(a, b) \subset \mathbb{R}$ is said to be *convex* if

$$f(\lambda x + (1-\lambda)y) \ \leq \ \lambda f(x) + (1-\lambda) f(y)$$

whenever $x, y \in (a, b)$ and $\lambda \in (0, 1)$. Prove that every convex function is continuous.

3. Let $\tau_n : S^n \to S^n$ be the antipodal map, and let X be the quotient of $S^n \times S^m$ bythe involution (τ_n, τ_m)—that is,

$$X \ = \ S^n \times S^m/(x, y) \sim (-x, -y) \ \forall \, (x, y) \, .$$

(a) What is the Euler characteristic of X?

(b) Find the homology groups of X in case $n = 1$.

4. Construct a surjective conformal mapping from the pie wedge

$$A = \{z = re^{i\theta} \ : \ \theta \in (0, \pi/4), \ r < 1\}$$

to the unit disk

$$D = \{z \ : \ |z| < 1\}.$$

5. Let $\mathbb{P} \cong \mathbb{P}^{mn-1}$ be the projective space of nonzero $m \times n$ matrices mod scalars, and let $M_k \subset \mathbb{P}$ be the locus of matrices of rank k or less.

(a) Show that M_k is an irreducible algebraic subvariety of $\mathbb{P}$.

(b) Find the dimension of M_k.

(c) In case $k = 1$, find the degree of M_1.

6. Compute the curvature and the torsion of the curve

$$\rho(t) = (t, t^2, t^3)$$

in $\mathbb{R}^3$.

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Wednesday September 19 2007 (Day 2)

1. Evaluate the integral

$$\int_0^\infty \frac{x^2}{x^4+5x^2+4}dx.$$

2. Consider the paraboloid $S \subset \mathbb{R}^3$ given by the equation $z = x^2 + y^2$. Let g be the metric on S induced by the one on $\mathbb{R}^3$.

(a) Write down the metric g in the coordinate system (x, y).

(b) Compute the Gaussian and the mean curvature of M.

3. Let D_5 denote the group of automorphisms of a regular pentagon. Let V be the 5 dimensional complex representation of D_5 corresponding to the action on the five edges of the pentagon. Decompose V as a sum of irreducible representations.

4. Consider the following three topological spaces:

$$A = \mathbb{CP}^3 \qquad B = S^2 \times S^4 \qquad \text{and} \qquad C = S^2 \vee S^4 \vee S^6$$

where $\mathbb{CP}^3$ is complex projective 3-space, S^n is an n-sphere and $\vee$ denotes connected sum.

(a) Calculate the cohomology groups (with integer coefficients) of all three

(b) Show that A and B are not homotopy equivalent

(c) Show that C is not homotopy equivalent to any compact manifold

5. Let $\mathcal{C}$ be the space $\mathcal{C}[0,1]$ with the sup norm $||f||_\infty$, and let $\mathcal{C}^1$ be the space $\mathcal{C}^1[0,1]$ with the sup norm $||f||_\infty + ||f'||_\infty$. Prove that the inclusion $\mathcal{C}^1 \subset \mathcal{C}$ is a compact operator.

6. Let K be a field of characteristic 0.

(a) Find two nonconstant rational functions $f(t), g(t) \in K(t)$ such that

$$f^2 = g^2 + 1.$$

(b) Now let n be any integer, $n \geq 3$. Show that there do not exist two nonconstant rational functions $f(t), g(t) \in K(t)$ such that

$$f^2 = g^n + 1.$$

QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Thursday September 20 2007 (Day 3)

1. Let R be the ring

$$R = \mathbb{C}[x, y, z]/(xy - z^2).$$

Find examples of the following ideals in R:

(a) a minimal prime ideal that is principal;

(b) a minimal prime ideal that is not principal;

(c) a maximal prime ideal than can be generated by two elements; and

(d) a maximal prime ideal than can not be generated by two elements

2. Find the Laurent expansion

$$f(z) = \sum_{n \in \mathbb{Z}} a_n z^n$$

around 0 of the function

$$f(z) = \frac{1}{z^2 - 3z + 2}$$

(a) valid in the open unit disc $\{z : |z| < 1\}$; and

(b) valid in the annulus $\{z : 1 < |z| < 2\}$.

3. (a) Show that any continuous map from the 2-sphere S^2 to a compact orientable 2-manifold of genus $g \geq 1$ is homotopic to a constant map.

(b) Recall that if $f : X \to Y$ is a map between compact, oriented n-manifolds, the induced map $f_* : H_n(X) \to H_n(Y)$ is multiplication by some integer d, called the *degree* of the map f. Now let S and T be compact oriented 2-manifolds of genus g and h respectively, and $f : S \to T$ a continuous map. Show that if $g > h$, then the degree of f is zero.

4. Let H be a (non-trivial) Hilbert space, and let $\mathcal{B}(H)$ denote the algebra of bounded linear operators on H. Recall that a linear operator $S : H \to H$ is called an adjoint to $T : H \to H$ if

$$(Tx, y) = (x, Sy) \tag{1}$$

holds for all $x, y \in H$.

(a) Prove that any $T \in \mathcal{B}(H)$ has a unique adjoint in $\mathcal{B}(H)$.

(b) Given $T \in \mathcal{B}(H)$, let T^* denote its adjoint. Prove that $(TS)^* = S^*T^*$ for $T, S \in \mathcal{B}(H)$.

(c) Prove that $\|Tx\| = \|T^*x\|$ for all $x \in H$ if and only if $TT^* = T^*T$.

(d) Prove that if $TT^* = T^*T$ then the eigenspaces corresponding to distinct eigenvalues of T are mutually orthogonal.

5. Prove that every group of order p^2q, where p and q are distinct primes, is solvable.

6. Let $\Gamma = \{p_1, \ldots, p_5\} \subset \mathbb{P}^2$ be a collection of five points in the plane.

(a) What is the Hilbert polynomial of the subvariety $\Gamma \subset \mathbb{P}^2$?

(b) How many different Hilbert functions can Γ have? List them all.