QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday September 16 2008 (Day 1)

1. (a) Prove that the Galois group G of the polynomial $X^6 + 3$ over $\mathbb{Q}$ is of order 6.

(b) Show that in fact G is isomorphic to the symmetric group S_3.

(c) Is there a prime number p such that $X^6 + 3$ is irreducible over the finite field of order p?

Solution. We initially work over any field k in which the polynomial $X^6 + 3$ is irreducible. Clearly k cannot have characteristic 2 or 3. Let α be a root of $X^6 + 3$ in an algebraic closure $\bar{k}$ of k, and set $\omega = (-1 + \alpha^3)/2$. Then a simple calculation gives $\omega^2 + \omega + 1 = 0$, so $\omega^3 = 1$ but $\omega \neq 1$. In fact, 1, ω, ω^2, -1, $-\omega$, $-\omega^2$ are all distinct elements of $\bar{k}$; they are the six roots of $X^6 + 1 = 0$, so α, $\omega\alpha$, $\omega^2\alpha$, $-\alpha$, $-\omega\alpha$, $-\omega^2\alpha$ are the six roots of $X^6 + 3 = 0$. These roots all lie in the extension $k(\alpha)$, which has degree 6 because α is a root of an irreducible degree 6 polynomial. So the Galois group of $X^6 + 3$ over k is of order 6.

The Galois group acts transitively on the roots of the polynomial $X^6 + 3$, so there are elements σ and τ of the Galois group sending α to $\omega\alpha$ and $-\alpha$ respectively. Then

$$\sigma(\omega) = \frac{-1 + \sigma(\alpha)^3}{2} = \frac{-1 + (\omega\alpha)^3}{2} = \frac{-1 + \alpha^3}{2} = \omega$$

and

$$\tau(\omega) = \frac{-1 + \tau(\alpha)^3}{2} = \frac{-1 + (-\alpha)^3}{2} = \frac{-1 - \alpha^3}{2} = -1 - \omega = \omega^2.$$

Therefore $\tau(\sigma(\alpha)) = \tau(\omega\alpha) = -\omega^2\alpha$ while $\sigma(\tau(\alpha)) = \sigma(-\alpha) = -\omega\alpha$, so σ and τ do not commute. So G is a nonabelian group of order 6, and thus must be isomorphic to the symmetric group S_3.

We now finish the problem.

(a) The polynomial $X^6 + 3$ is irreducible over $\mathbb{Q}$ by Eisenstein's criterion at the prime 3. So the preceding arguments show that the Galois group of $X^6 + 3$ over $\mathbb{Q}$ is of order 6.

(b) Similarly, we also showed under the same assumption that the Galois group was isomorphic to S_3.

(c) No, there is no prime p such that X^6+3 is irreducible over the finite field of order p. If there was, then by the preceding arguments, the extension formed by adjoining a root of X^6+3 would be a Galois extension with Galois group S_3. But the Galois groups of finite extensions of the field of order p are all cyclic groups, a contradiction.

2. Evaluate the integral
$$\int_0^\infty \frac{\sqrt{t}}{(1+t)^2} dt.$$

Solution. Write $\sqrt{z}$ for the branch of the square root function defined on $\mathbb{C}-[0,\infty)$ such that $\sqrt{z}$ has positive real part when $z=r+\epsilon i$, ϵ small and positive. Using the identity $(\sqrt{z})^2=z$ one can check that $\frac{d\sqrt{z}}{dz}=\frac{1}{2\sqrt{z}}$.

Define the meromorphic function f on $\mathbb{C}-[0,\infty)$ by $f(z)=\sqrt{z}/(1+z)^2$. Let $\epsilon>0$ be small and R large, and let γ be the contour which starts at ϵi, travels along the ray $z=[0,\infty)+\epsilon i$ until it reaches the circle $|z|=R$, traverses most of that circle counterclockwise stopping at the ray $z=[0,\infty)-\epsilon i$, then travels along that ray backwards, and finally traverses the semicircle $|z|=\epsilon$ in the left half-plane to get back to ϵi. Consider the contour integral $\int_\gamma f(z)\,dz$. The contribution from the first ray is approximately the desired integral $I=\int_0^\infty \sqrt{t}/(1+t^2)\,dt$; the contribution from the large circle is small, because when $|z|=R$, $|\sqrt{z}/(1+z)^2|$ is about $R^{-3/2}$, and the perimeter of the circle is only about $2\pi R$; the contribution from the second ray is about I again, because the sign from traveling in the opposite direction cancels the sign coming from the branch cut in $\sqrt{z}$; and the contribution from the small circle is small because $f(z)$ is bounded in a neighborhood of 0. So

$$2I=\lim_{\epsilon\to 0,R\to\infty}\int_\gamma \frac{\sqrt{z}}{(1+z)^2}\,dz = 2\pi i\left.\frac{d\sqrt{z}}{dz}\right|_{z=-1} = 2\pi i\frac{1}{2\sqrt{-1}}=\pi$$

and thus $I=\pi$.

3. For $X\subset\mathbb{R}^3$ a smooth oriented surface, we define the *Gauss map* $g: X\to S^2$ to be the map sending each point $p\in X$ to the unit normal vector to X at p. We say that a point $p\in X$ is *parabolic* if the differential $dg_p: T_p(X)\to T_{g(p)}(S^2)$ of the map g at p is singular.

(a) Find an example of a surface X such that every point of X is parabolic.

(b) Suppose now that the locus of parabolic points is a smooth curve $C\subset X$, and that at every point $p\in C$ the tangent line $T_p(C)\subset T_p(X)$ coincides with the kernel of the map dg_p. Show that C is a planar curve, that is, each connected component lies entirely in some plane in $\mathbb{R}^3$.

Solution.

(a) Let X be the xy-plane; then the Gauss map $g : X \to S^2$ is constant, so its differential is everywhere zero and hence singular.

(b) Consider the Gauss map of X restricted to C, $g|_C : C \to S^2$. Then for any point $p \in C$, $d(g|_C)_p = (dg_p)|_{T_p(C)}$, which is 0 by assumption. Hence $g|_C$ is locally constant on C. That is, on each connected component C_0 of C there is a fixed vector (the value of $g|_C$ at any point of the component) normal to all of C_0. Hence C_0 lies in a plane in $\mathbb{R}^3$ normal to this vector.

4. Let $X = (S^1 \times S^1) \setminus \{p\}$ be a once-punctured torus.

(a) How many connected, 3-sheeted covering spaces $f : Y \to X$ are there?

(b) Show that for any of these covering spaces, Y is either a 3-times punctured torus or a once-punctured surface of genus 2.

Solution.

(a) By covering space theory, the number of connected, 3-sheeted covering spaces of a space Z is the number of conjugacy classes of subgroups of index 3 in the fundamental group $\pi_1(Z)$. (We consider two covering spaces of Z isomorphic only when they are related by an homeomorphism over the identity on Z, not one over any homeomorphism of Z.) So we may replace X by the homotopy equivalent space $X' = S^1 \vee S^1$. If we view this new space X' as a graph with one vertex and two directed loops labeled a and b, then a connected 3-sheeted cover of X' is a connected graph with three vertices and some directed edges labeled a or b such that each vertex has exactly one incoming and one outgoing edge with each of the labels a and b. Temporarily treating the three vertices as having distinct labels x, y, z, we find six ways the a edges can be placed: loops at x, y and z; a loop at x and edges from y to z and from z to y; similarly but with the loop at y; similarly but with the loop at z; edges from x to y, y to z, and z to x; and edges from x to z, z to y, and y to x. Analogously there are six possible placements for the b edges. Considering all possible combinations, throwing out the disconnected ones, and then treating two graphs as the same if they differ only in the labels x, y, z, we arrive at seven distinct possibilities.

(b) Let C be a small loop in $S^1 \times S^1$ around the removed point p, and let $X_0 \subset X$ be the torus with the interior of C removed, so that X_0 is a compact manifold with boundary $C = S^1$. Now let Y be any connected, 3-sheeted covering space of X. Pull back the covering map $Y \to X$ along the inclusion $X_0 \to X$ to obtain a 3-sheeted covering space Y_0 of X_0. Since $X_0 \to X$ is a homotopy equivalence, so is $Y_0 \to Y$ and in particular Y_0 is still connected. We can recover Y from Y_0 by gluing a strip $D \times [0, 1)$ along the preimage D of C in Y_0. So, it will suffice to show that Y is either a torus with three small disks removed, or a surface of genus two with one small disk removed.

Since Y_0 is a 3-sheeted cover of X_0, it is a compact oriented surface with boundary. By the classification of compact oriented surfaces with boundary, Y_0 can be formed by taking a surface of some genus g and removing some number d of small disks. The boundary of Y_0 is D, the preimage of C, which is a (not necessarily connected) 3-sheeted cover of C. So Y_0 has either one or three boundary circles, i.e., $d = 1$ or $d = 3$. Moreover, we can compute using the Euler characteristic that

$$2 - 2g - d = \chi(Y_0) = 3\chi(X_0) = -3.$$

If $d = 3$, then $g = 1$; if $d = 1$, then $g = 2$. So Y is correspondingly either a 3-times punctured torus or a once-punctured surface of genus two.

5. Let X be a complete metric space with metric ρ. A map $f : X \to X$ is said to be *contracting* if for any two distinct points $x, y \in X$,

$$\rho(f(x), f(y)) < \rho(x, y).$$

The map f is said to be *uniformly contracting* if there exists a constant $c < 1$ such that for any two distinct points $x, y \in X$,

$$\rho(f(x), f(y)) < c \cdot \rho(x, y).$$

(a) Suppose that f is uniformly contracting. Prove that there exists a unique point $x \in X$ such that $f(x) = x$.

(b) Give an example of a contracting map $f : [0, \infty) \to [0, \infty)$ such that $f(x) \neq x$ for all $x \in [0, \infty)$.

Solution.

(a) We first show there exists at least one fixed point of f. Let $x_0 \in X$ be arbitrary and define a sequence x_1, x_2, ..., by $x_n = f(x_{n-1})$. Let $d = \rho(x_0, x_1)$. By the uniformly contracting property of f, $\rho(x_n, x_{n+1}) \leq dc^n$ for every n. Now observe

$$\begin{aligned}\rho(x_n, x_{n+k}) &\leq \rho(x_n, x_{n+1}) + \cdots + \rho(x_{n+k-1}, x_{n+k}) \\ &\leq dc^n + \cdots + dc^{n+k-1} \\ &\leq dc^n/(1-c).\end{aligned}$$

This expression tends to 0 as n increases, so (x_n) is a Cauchy sequence and thus has a limit x by the completeness of X. Now f is continuous, because it is uniformly contracting, so

$$f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x,$$

and x is a fixed point of f, as desired.

To show that f has at most one fixed point, suppose x and y were distinct points of X with $f(x) = x$ and $f(y) = y$. Then

$$\rho(x, y) = \rho(f(x), f(y)) < c\rho(x, y),$$

which is impossible since $\rho(x, y) > 0$ and $c < 1$.

(b) Let $f(x) = x + e^{-x}$. Then $f'(x) = 1 - e^{-x} \in [0, 1)$ for all $x \geq 0$, so by the Mean Value Theorem $0 \leq f(x) - f(y) < x - y$ for any $x > y \geq 0$. Thus f is contracting. But f has no fixed points, because $x + e^{-x/2} \neq x$ for all x.

6. Let K be an algebraically closed field of characteristic other than 2, and let $Q \subset \mathbb{P}^3$ be the surface defined by the equation

$$X^2 + Y^2 + Z^2 + W^2 = 0.$$

(a) Find equations of all lines $L \subset \mathbb{P}^3$ contained in Q.

(b) Let $\mathbb{G} = \mathbb{G}(1, 3) \subset \mathbb{P}^5$ be the Grassmannian of lines in $\mathbb{P}^3$, and $F \subset \mathbb{G}$ the set of lines contained in Q. Show that $F \subset \mathbb{G}$ is a closed subvariety.

Solution.

(a) Since K is algebraically closed and of characteristic other than 2, we may replace the quadratic form $X^2 + Y^2 + Z^2 + W^2$ with any other nondegenerate one, such as $AB + CD$. More explicitly, set $A = X + \sqrt{-1}Y$, $B = X - \sqrt{-1}Y$, $C = Z + \sqrt{-1}W$, $D = -Z + \sqrt{-1}W$; this change of coordinates is invertible because we can divide by 2, and $AB - CD = X^2 + Y^2 + Z^2 + W^2$.

A line contained in the surface in $\mathbb{P}^3$ defined by $AB - CD = 0$ is the same as a plane in the subset of the vector space K^4 defined by $v_1v_2 - v_3v_4 = 0$. Define a bilinear form $(\cdot, \cdot)$ on K^4 by $(v, w) = v_1w_2 + v_2w_1 - v_3w_4 - v_4w_3$. Then we want to find all the planes $V \subset K^4$ such that $(v, v) = 0$ for every $v \in V$. Observe that

$$(v + w, v + w) - (v, v) - (w, w) = (v, w) + (w, v) = 2(v, w),$$

so it is equivalent to require that $(v, w) = 0$ for all v and $w \in V$.

Suppose now that V is such a plane inside K^4. Then V has nontrivial intersection with the subspace $\{v_1 = 0\}$; let $v \in V$ be a nonzero vector with $v_1 = 0$. Since $v_1v_2 - v_3v_4 = 0$, we must have either $v_3 = 0$ or $v_4 = 0$. Assume without loss of generality that $v_3 = 0$. Write $u = v_2$, $t = v_4$; then $(u, t) \neq (0, 0)$. Now consider any vector $w \in V$; then

$$0 = (w, v) = w_1v_2 + w_2v_1 - w_3v_4 - w_4v_3 = uw_1 - tw_3.$$

So there exists $r \in K$ such that $w_1 = rt$ and $w_3 = ru$. We also have

$$0 = \frac{1}{2}(w, w) = w_1 w_2 - w_3 w_4 = rtw_2 - ruw_4.$$

Hence either $r = 0$ or there exists $s \in K$ such that $w_2 = su$ and $w_4 = st$. So

$$V \subset \{ (w_1, 0, w_3, 0) \mid w_1, w_3 \in K \} \cup \{ (rt, su, ru, st) \mid r, s \in K \}.$$

Since V has dimension 2, we conclude that V must be equal to one of these two planes.

This discussion was under the assumption that $v_3 = 0$ rather than $v_4 = 0$; in the second case, we find that V is of one of the forms $\{ (w_1, 0, 0, w_4) \mid w_1, w_4 \in K \}$ or $\{ (rt, su, st, ru) \mid r, s \in K \}$ for $(u, t) \neq (0, 0)$. But we obtain $\{ (w_1, 0, w_3, 0) \mid w_1, w_3 \in K \}$ by setting $(u, t) = (0, 1)$ in $\{ (rt, su, st, ru) \mid r, s \in K \}$ and $\{ (w_1, 0, 0, w_4) \mid w_1, w_4 \in K \}$ by setting $(u, t) = (0, 1)$ in $\{ (rt, su, ru, st) \mid r, s \in K \}$. Hence all such planes V are of one of the forms

$$V^{(1)}_{u,t} = \{ (rt, su, ru, st) \mid r, s \in K \}$$

or

$$V^{(2)}_{u,t} = \{ (rt, su, st, ru) \mid r, s \in K \}$$

for some $(u, t) \neq (0, 0)$. And it is easy to see conversely that each of these subspaces is two-dimensional and lies in the subset of K^4 determined by $(v, v) = 0$.

Translating this back into equations for the lines on the surface Q, we obtain two families of lines:

$$L^{(1)}_{u,t} = \left\{ \left[\frac{rt + su}{2} : \frac{rt - su}{2\sqrt{-1}} : \frac{ru - st}{2} : \frac{ru + st}{2\sqrt{-1}} \right] \mid r, s \in K \right\},$$

$$L^{(2)}_{u,t} = \left\{ \left[\frac{rt + su}{2} : \frac{rt - su}{2\sqrt{-1}} : \frac{st - ru}{2} : \frac{st + ru}{2\sqrt{-1}} \right] \mid r, s \in K \right\},$$

where (u, t) ranges over $K^2 \setminus \{(0, 0)\}$. The families $L^{(1)}_{*,*}$ and $L^{(2)}_{*,*}$ are disjoint, and two pairs (u, t) and (u', t') yield the same line in a given family if and only if one pair is a nonzero scalar multiple of the other.

(b) By the result of the previous part, F is the image of a regular map $\mathbb{P}^1 \amalg \mathbb{P}^1 \to \mathbb{G}$, so F is a closed subvariety of $\mathbb{G}$.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Wednesday September 17 2008 (Day 2)

1. (a) Show that the ring $\mathbb{Z}[i]$ is Euclidean.

(b) What are the units in $\mathbb{Z}[i]$?

(c) What are the primes in $\mathbb{Z}[i]$?

(d) Factorize $11 + 7i$ into primes in $\mathbb{Z}[i]$.

Solution.

(a) We define a norm on $\mathbb{Z}[i]$ in the usual way, $|a+bi| = \sqrt{a^2+b^2}$. Then we must show that for any a and b in $\mathbb{Z}[i]$ with $b \neq 0$, there exist q and r in $\mathbb{Z}[i]$ with $a = qb + r$ and $|r| < |b|$. Let $q_0 = a/b \in \mathbb{C}$ and let $q \in \mathbb{Z}[i]$ be one of the Gaussian integers closest to q_0; the real and imaginary parts of q differ by at most $\frac{1}{2}$ from those of q_0, so $|q - q_0| \leq \sqrt{2}/2 < 1$. Now let $r = a - qb$. Then

$$|r| = |a - qb| = |(q_0 - q)b| = |q_0 - q||b| < |b|$$

as desired.

(b) If $u \in \mathbb{Z}[i]$ is a unit, then there exists $u' \in \mathbb{Z}[i]$ such that $uu' = 1$, so $|u||u'| = 1$ and hence $|u| = 1$ (since $|z| > 0$ for every $z \in \mathbb{Z}[i]$). Writing $u = a + bi$, we obtain $1 = |u| = \sqrt{a^2+b^2}$ so either $a = \pm 1$ and $b = 0$ or $a = 0$ and $b = \pm 1$. The four possibilities $u = 1, -1, i, -i$ are all clearly units.

(c) Since $\mathbb{Z}[i]$ is Euclidean, it contains a greatest common divisor of any two elements, and it follows that irreducibles and primes are the same: if z is irreducible and $z \nmid x$ and $n \nmid y$, then $\gcd(x, z) = \gcd(y, z) = 1$, so $1 \in (x, z)$ and $1 \in (y, z)$; hence $1 \in (xy, z)$, so $z \nmid xy$.

Let $z \in \mathbb{Z}[i]$. If $|z| \leq 1$, then z is either zero or a unit so is not prime. If $|z| = \sqrt{p}$, $p \in \mathbb{Z}$ a prime, then u must be a prime in $\mathbb{Z}[i]$, because $|\cdot|$ is multiplicative and $|z|^2 \in \mathbb{Z}$ for all $z \in \mathbb{Z}[i]$. It remains to consider z for which $|z|^2$ is composite.

Write $\sqrt{N} = |z|$, and factor $N = p_1 p_2 \cdots p_r$ in $\mathbb{Z}$. Note that

$$z \mid z\bar{z} = N = p_1 p_2 \cdots p_r$$

so if z is prime, then z divides one of the primes $p = p_i$ in $\mathbb{Z}[i]$. Moreover $\bar{z}$ also divides p so $N = z\bar{z}$ divides p^2; since N is composite we must have $N = p^2$. That is, $z\bar{z} = p^2$; by assumption the left side is a factorization into irreducibles, so up to units each p on the right hand side must be a product of some terms on the left; the only possibility is $z = pu$, $\bar{z} = p\bar{u}$ for some unit u. Now when $p \equiv 3 \pmod 4$, p is indeed a prime in $\mathbb{Z}[i]$, because then $p \mid a^2 + b^2 \implies p \mid a, b \implies p^2 \mid a^2 + b^2$, so there are no elements of $\mathbb{Z}[i]$ with norm $\sqrt{p}$. If $p \equiv 1 \pmod 4$, then p can be written in the form $p = a^2 + b^2$, so $p = (a + bi)(a - bi)$ and p is not in fact a prime.

In conclusion, the primes of $\mathbb{Z}[i]$ are

- elements $z \in \mathbb{Z}[i]$ with $z = \sqrt{p}$, $p \in \mathbb{Z}$ prime (necessarily congruent to 1 mod 4);
- elements of the form pu with $p \in \mathbb{Z}$ a prime congruent to 3 mod 4 and $u \in \mathbb{Z}[i]$ a unit.

(d) We first compute $|11 + 7i| = \sqrt{121 + 49} = \sqrt{170}$; so $11 + 7i$ will be a product of primes with norms $\sqrt{2}$, $\sqrt{5}$ and $\sqrt{17}$. There is only one prime with norm $\sqrt{2}$ up to units and only two with a norm $\sqrt{5}$; a quick calculation yields

$$11 + 7i = (1 + i)(1 + 2i)(1 - 4i).$$

2. Let $U \subset \mathbb{C}$ be the open region

$$U = \{z : |z - 1| < 1 \text{ and } |z - i| < 1\}.$$

Find a conformal map $f : U \to \Delta$ of U onto the unit disc $\Delta = \{z : |z| < 1\}$.

Solution. The map $z \mapsto 1/z$ takes the open discs $\{ z : |z - 1| < 1 \}$ and $\{ z : |z - i| < 1 \}$ holomorphically to the open half-planes $\{ z : \Re z \geq \frac{1}{2} \}$ and $\{ z : \Im z \leq -\frac{1}{2} \}$ respectively, so it takes U to their intersection. So we can define a conformal isomorphism f_0 from U to the interior U' of the fourth quadrant by

$$f_0(z) = \frac{1}{z} - \frac{1 - i}{2}.$$

Now we can send U' to the lower half plane by the squaring map, and that to Δ by the Möbius transformation $z \mapsto \frac{1}{z - i/2} - i$. Thus the composite

$$\frac{1}{(\frac{1}{z} - \frac{1-i}{2})^2 + \frac{i}{2}} - i$$

is actually a conformal isomorphism from U to Δ.

3. Let n be a positive integer, A a symmetric $n \times n$ matrix and Q the quadratic form

$$Q(x) = \sum_{1 \leq i,j \leq n} A_{i,j} x_i x_j.$$

Define a metric on $\mathbb{R}^n$ using the line element whose square is

$$ds^2 = e^{Q(x)} \sum_{1 \leq i \leq n} dx^i \otimes dx^i.$$

(a) Write down the differential equation satisfied by the geodesics of this metric

(b) Write down the Riemannian curvature tensor of this metric at the origin in $\mathbb{R}^n$.

Solution. We first compute the Christoffel symbols $\Gamma^m{}_{ij}$ with respect to the standard basis for the tangent space $(\partial/\partial x_k$. The metric tensor in these coordinates is

$$g_{ij} = \delta_{ij} e^{Q(x)} \quad \text{with inverse} \quad g^{ij} = \delta_{ij} e^{-Q(x)}.$$

Its partial derivatives are

$$\frac{\partial}{\partial x_k} g_{ij} = \delta_{ij} e^{Q(x)} \frac{\partial}{\partial x_k} Q(x) = 2\delta_{ij} e^{Q(x)} \sum_l A_{lk} x_l.$$

Then (using implicit summation notation)

$$\begin{aligned}
\Gamma^m{}_{ij} &= \frac{1}{2} g^{km} \left(\frac{\partial}{\partial x_i} g_{kj} + \frac{\partial}{\partial x_j} g_{ik} - \frac{\partial}{\partial x_k} g_{ij} \right) \\
&= \frac{1}{2} \delta_{km} e^{-Q(x)} (2\delta_{kj} e^{Q(x)} A_{li} x_l + 2\delta_{ik} e^{Q(x)} A_{lj} x_l - 2\delta_{ij} e^{Q(x)} A_{lk} x_l) \\
&= (\delta_{mj} A_{li} + \delta_{im} A_{lj} - \delta_{ij} A_{lm}) x_l.
\end{aligned}$$

(a) The geodesic equation is

$$\begin{aligned}
0 &= \frac{d^2 x_m}{dt^2} + \Gamma^m{}_{ij} \frac{dx_i}{dt} \frac{dx_j}{dt} \\
&= \frac{d^2 x_m}{dt^2} + (\delta_{mj} A_{li} + \delta_{im} A_{lj} - \delta_{ij} A_{lm}) x_l \frac{dx_i}{dt} \frac{dx_j}{dt} \\
&= \frac{d^2 x_m}{dt^2} + 2 \sum_{i,l} A_{li} x_l \frac{dx_i}{dt} \frac{dx_m}{dt} - \sum_l A_{lm} x_l \sum_i \left(\frac{dx_i}{dt} \right)^2
\end{aligned}$$

(where we have written summations explicitly on the last line).

(b) The Riemannian curvature tensor is given by

$$\begin{aligned}
R^l{}_{ijk} &= \frac{\partial}{\partial x_j} \Gamma^l{}_{ik} - \frac{\partial}{\partial x_k} \Gamma^l{}_{ij} + \Gamma^l{}_{js} \Gamma^s{}_{ik} - \Gamma^l{}_{ks} \Gamma^s_{ij} \\
&= (\delta_{lk} A_{ri} + \delta_{il} A_{rk} - \delta_{ik} A_{rl}) - (\delta_{lj} A_{ri} + \delta_{il} A_{rj} - \delta_{ij} A_{rl}) \\
&\quad + (\delta_{ls} A_{tj} + \delta_{jl} A_{ts} - \delta_{js} A_{tl}) x_t (\delta_{sk} A_{ui} + \delta_{is} A_{uk} - \delta_{ik} A_{us}) x_u \\
&\quad - (\delta_{ls} A_{tk} + \delta_{kl} A_{ts} - \delta_{ks} A_{tl}) x_t (\delta_{sj} A_{ui} + \delta_{is} A_{uj} - \delta_{ij} A_{us}) x_u.
\end{aligned}$$

4. Let H be a separable Hilbert space and $b : H \to H$ a bounded linear operator.

(a) Prove that there exists $r > 0$ such that $b + r$ is invertible.

(b) Suppose that H is infinite dimensional and that b is compact. Prove that b is not invertible.

Solution.

(a) It is equivalent to show that there exists $\epsilon > 0$ such that $1-\epsilon b$ is invertible. Since b is bounded there is a constant C such that $||bv|| \leq C||v||$ for all $v \in H$. Choose $\epsilon < 1/C$ and consider the series

$$a = 1 + \epsilon b + \epsilon^2 b^2 + \cdots .$$

For any v the sequence $v + \epsilon bv + \epsilon^2 b^2 v + \cdots$ converges by comparison to a geometric series. So this series converges to a linear operator a and $a(1 - \epsilon b) = (1 - \epsilon b)a = 1$, that is, $a = (1 - \epsilon b)^{-1}$.

(b) Suppose for the sake of contradiction that b is invertible. Then the open mapping theorem applies to b, so if $U \subset H$ is the unit ball, then $b(U)$ contains the ball around 0 of radius ε for some $\varepsilon > 0$. By the definition of a compact operator, the closure V of $b(U)$ is a compact subset of H. But H is infinite dimensional, so there is an infinite orthonormal set v_1, v_2, ..., and the sequence εv_1, εv_2, ... is contained in V but has no limit point, a contradiction. Hence b cannot be invertible.

5. Let $X \subset \mathbb{P}^n$ be a projective variety.

(a) Define the *Hilbert function* $h_X(m)$ and the *Hilbert polynomial* $p_X(m)$ of X.

(b) What is the significance of the degree of p_X? Of the coefficient of its leading term?

(c) For each m, give an example of a variety $X \subset \mathbb{P}^n$ such that $h_X(m) \neq p_X(m)$.

Solution.

(a) The homogeneous coordinate ring $S(X)$ is the graded ring $S(\mathbb{P}^n)/I$, where $S(\mathbb{P}^n)$ is the ring of polynomials in $n+1$ variables and I is the ideal generated by those homogeneous polynomials which vanish on X. Then $h_X(m)$ is the dimension of the mth graded piece of this ring. The Hilbert polynomial $p_X(m)$ is the unique polynomial such that $p_X(m) = h_X(m)$ for all sufficiently large integers m.

(b) The degree of p_X is the dimension d of the variety $X \subset \mathbb{P}^n$, and its leading term is $\deg X/d!$.

(c) Let X consist of any k distinct points of $\mathbb{P}^n$. Then X is a variety of dimension 0 and degree k, so by the previous part $p_X(m) = k$. But $h_X(m)$ is at most the dimension of the space of homogeneous degree m polynomials in $n + 1$ variables, so for sufficiently large k, $h_X(m) < k = p_X(m)$.

6. Let $X = S^2 \vee \mathbb{RP}^2$ be the wedge of the 2-sphere and the real projective plane. (This is the space obtained from the disjoint union of the 2-sphere and the real projective plane by the equivalence relation that identifies a given point in S^2 with a given point in $\mathbb{RP}^2$, with the quotient topology.)

(a) Find the homology groups $H_n(X, \mathbb{Z})$ for all n.

(b) Describe the universal covering space of X.

(c) Find the fundamental group $\pi_1(X)$.

Solution.

(a) The wedge $A \vee B$ of two spaces satisfies $\tilde{H}_n(A \vee B, \mathbb{Z}) = \tilde{H}_n(A, \mathbb{Z}) \oplus \tilde{H}_n(B, \mathbb{Z})$ for all n, so

$$H_0(X, \mathbb{Z}) = \mathbb{Z}, \quad H_1(X, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}, \quad H_2(X, \mathbb{Z}) = \mathbb{Z}.$$

(b) The universal covering space $\tilde{X}$ of X can be constructed as the union of the unit spheres centered at $(-2, 0, 0)$, $(0, 0, 0)$ and $(2, 0, 0)$ in $\mathbb{R}^3$; the group $\mathbb{Z}/2\mathbb{Z}$ acts freely on $\tilde{X}$ by sending x to $-x$, and the quotient is X. Topologically, $\tilde{X}$ is the wedge sum $S^2 \vee S^2 \vee S^2$.

(c) Since X is the quotient of the simply connected space $\tilde{X}$ by a free action of the group $\mathbb{Z}/2\mathbb{Z}$, we have $\pi_1(X) = \mathbb{Z}/2\mathbb{Z}$.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Thursday January 31 2008 (Day 3)

1. For $z \in \mathbb{C} \setminus \mathbb{Z}$, set

$$f(z) = \lim_{N\to\infty} \left(\sum_{n=-N}^{N} \frac{1}{z+n} \right)$$

(a) Show that this limit exists, and that the function f defined in this way is meromorphic.

(b) Show that $f(z) = \pi \cot \pi z$.

Solution.

(a) We can rewrite f as

$$f(z) = \frac{1}{z} + \lim_{N\to\infty} \left(\sum_{n=1}^{N} \frac{1}{z+n} + \frac{1}{z-n} \right) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}.$$

For any $z \in \mathbb{C} \setminus \mathbb{R}$, the terms of this sum are uniformly bounded near z by a convergent series. So this sum of analytic functions converges uniformly near z and thus f is analytic near z. We can apply a similar argument to $f(z) - \frac{1}{z-n}$ to conclude that f has a simple pole at each integer n (with residue 1).

(b) The meromorphic function $\pi \cot \pi z$ also has a simple pole at each integer n with residue $\lim_{z\to n}(z-n)(\pi \cot \pi z) = 1$, so $f(z) - \pi \cot \pi z$ is a global analytic function. Moreover

$$\begin{aligned} f(z+1) - f(z) &= \lim_{N\to\infty} \left(\sum_{n=-N}^{N} \frac{1}{z+1+n} - \frac{1}{z+n} \right) \\ &= \lim_{N\to\infty} \left(\frac{1}{z+1+N} - \frac{1}{z-N} \right) \\ &= 0 \end{aligned}$$

for all $z \in \mathbb{C} \setminus \mathbb{Z}$, and $\cot \pi(z+1) = \cot \pi z$, so $f(z) - \pi \cot \pi z$ is periodic with period 1. Its derivative is

$$f'(z) - \frac{d}{dz}\pi \cot \pi z = -\frac{1}{z^2} + \sum_{n=1}^{\infty} \left(-\frac{1}{(z+n)^2} - \frac{1}{(z-n)^2} \right) + \pi^2 \sin^2 \pi z.$$

This is again an analytic function with period 1, and it approaches 0 as the imaginary part of z goes to ∞, so it must be identically 0. So

$f(z) - \pi \cot \pi z$ is constant; since it is an odd function, that constant must be 0.

2. Let p be an odd prime.

(a) What is the order of $GL_2(\mathbb{F}_p)$?

(b) Classify the finite groups of order p^2.

(c) Classify the finite groups G of order p^3 such that every element has order p.

Solution.

(a) To choose an invertible 2×2 matrix over $\mathbb{F}_p$, we first choose its first column to be any nonzero vector in $p^2 - 1$, then its second column to be any vector not a multiple of the first in $p^2 - p$ ways. So $GL_2(\mathbb{F}_p)$ has $(p^2 - 1)(p^2 - p)$ elements.

(b) Let G be a group with p^2 elements. As a p-group, G must have nontrivial center Z. If $Z = G$, then G is abelian and so $G = (\mathbb{Z}/p\mathbb{Z})^2$ or $G = \mathbb{Z}/p^2\mathbb{Z}$. Otherwise Z has order p. So there is a short exact sequence

$$1 \to Z \to G \to \mathbb{Z}/p\mathbb{Z} \to 1.$$

The sequence splits, because we can pick a generator for $\mathbb{Z}/p\mathbb{Z}$ and choose a preimage for it in G; this preimage has order p (G cannot contain an element of order p^2 or it would be cyclic) so it determines a splitting $\mathbb{Z}/p\mathbb{Z} \to G$. Hence G is the direct product of Z and $\mathbb{Z}/p\mathbb{Z}$ (because Z is central in G). So there are no new groups in this case.

(c) Let G be a group with p^3 elements in which every element has order p, and let Z be the center of G; again Z is nontrivial. If Z has order p^3, then G is abelian, and since every element has order p, G must be $(\mathbb{Z}/p\mathbb{Z})^3$. If Z has order p^2, then Z must be isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$, and there is a short exact sequence

$$1 \to Z \to G \to \mathbb{Z}/p\mathbb{Z} \to 1.$$

Again, we can split this sequence by choosing a preimage of a generator of $\mathbb{Z}/p\mathbb{Z}$, so G is the direct product $Z \times \mathbb{Z}/p\mathbb{Z}$. Hence Z is not really the center of G, and there are no groups in this case. Finally, suppose Z has order p; then there is a short exact sequence

$$1 \to Z \to G \to (\mathbb{Z}/p\mathbb{Z})^2 \to 1.$$

Let a and b be elements of G whose images together generate $(\mathbb{Z}/p\mathbb{Z})^2$. Then the image of $c = bab^{-1}a^{-1}$ is $0 \in (\mathbb{Z}/p\mathbb{Z})^2$, so c lies in Z. If a and b commuted, we could split this sequence which would lead to a contradiction as before. Hence c is a generator of Z. We can write every element of G uniquely in the form $a^i b^j c^k$ with $0 \le i,\ j,\ k < p$, and we know the commutation relations between a, b and c; it's easy to see that G is isomorphic to the group of upper-triangular 3×3 matrices over $\mathbb{F}_p$ with ones on the diagonal via the isomorphism

$$a^i b^j c^k \leftrightarrow \begin{pmatrix} 1 & j & k \\ 0 & 1 & i \\ 0 & 0 & 1 \end{pmatrix}.$$

It remains to check that in this group every element really has order p. But one can check by induction that

$$\begin{pmatrix} 1 & j & k \\ 0 & 1 & i \\ 0 & 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & nj & nk + \frac{n(n-1)}{2} ij \\ 0 & 1 & ni \\ 0 & 0 & 1 \end{pmatrix}$$

and setting $n = p$, the right hand side is the identity because p is odd.

3. Let X and Y be compact, connected, oriented 3-manifolds, with

$$\pi_1(X) = (\mathbb{Z}/3\mathbb{Z}) \oplus \mathbb{Z} \oplus \mathbb{Z} \qquad \text{and} \qquad \pi_1(Y) = (\mathbb{Z}/6\mathbb{Z}) \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

(a) Find $H_n(X, \mathbb{Z})$ and $H_n(Y, \mathbb{Z})$ for all n.

(b) Find $H_n(X \times Y, \mathbb{Q})$ for all n.

Solution.

(a) (We omit the coefficient group $\mathbb{Z}$ from the notation in this part.) By the Hurewicz theorem, $H_1(X)$ is the abelianization of $\pi_1(X)$, so $H_1(X) = (\mathbb{Z}/3\mathbb{Z}) \oplus \mathbb{Z} \oplus \mathbb{Z}$. By Poincaré duality, $H^2(X) = (\mathbb{Z}/3\mathbb{Z}) \oplus \mathbb{Z} \oplus \mathbb{Z}$ as well. Now by the universal coefficient theorem for cohomology, $H^1(X)$ is (noncanonically isomorphic to) the free part of $H_1(X)$. So $H^1(X) = \mathbb{Z} \oplus \mathbb{Z}$, and by Poincaré duality again $H_2(X) = \mathbb{Z} \oplus \mathbb{Z}$ too. Of course, $H_3(X) = \mathbb{Z}$ because X is a connected oriented 3-manifold. So the homology groups of X are

$$H_0(X) = \mathbb{Z}, \quad H_1(X) = (\mathbb{Z}/3\mathbb{Z}) \oplus \mathbb{Z}^2, \quad H_2(X) = \mathbb{Z}^2, \quad H_3(X) = \mathbb{Z}.$$

Entirely analogous arguments for Y yield

$$H_0(Y) = \mathbb{Z}, \quad H_1(Y) = (\mathbb{Z}/6\mathbb{Z}) \oplus \mathbb{Z}^3, \quad H_2(Y) = \mathbb{Z}^3, \quad H_3(Y) = \mathbb{Z}.$$

(b) The module $\mathbb{Q}$ is flat over $\mathbb{Z}$ ($\mathrm{Tor}_n^{\mathbb{Z}}(\mathbb{Q}, -) = 0$ for $n > 0$) so for any space A, $H_n(A, \mathbb{Q}) = \mathbb{Q} \otimes H_n(A, \mathbb{Z})$. In particular,

$$H_0(X, \mathbb{Q}) = \mathbb{Q}, \quad H_1(X, \mathbb{Q}) = \mathbb{Q}^2, \quad H_2(X, \mathbb{Q}) = \mathbb{Q}^2, \quad H_3(X, \mathbb{Q}) = \mathbb{Q},$$

$$H_0(Y,\mathbb{Q}) = \mathbb{Q}, \quad H_1(Y,\mathbb{Q}) = \mathbb{Q}^3, \quad H_2(Y,\mathbb{Q}) = \mathbb{Q}^3, \quad H_3(Y,\mathbb{Q}) = \mathbb{Q}.$$

The Künneth theorem over a field k states that $H_*(A\times B, k) = H_*(A,k)\otimes H_*(B,k)$ for any spaces A and B. So the homology groups $H_n(X\times Y,\mathbb{Q})$ for $n = 0, \ldots, 6$ are

$$\mathbb{Q}, \quad \mathbb{Q}^5, \quad \mathbb{Q}^{11}, \quad \mathbb{Q}^{14}, \quad \mathbb{Q}^{11}, \quad \mathbb{Q}^5, \quad \mathbb{Q}.$$

Note. Actually, there are no compact connected 3-manifolds M with $\pi_1(M) = (\mathbb{Z}/3\mathbb{Z})\oplus\mathbb{Z}\oplus\mathbb{Z}$ or $\pi_1(M) = (\mathbb{Z}/6\mathbb{Z})\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus Z$. The only abelian groups which are the fundamental groups of compact connected 3-manifolds are $\mathbb{Z}/n\mathbb{Z}$, $\mathbb{Z}$, $\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$, and $(\mathbb{Z}/2\mathbb{Z})\oplus\mathbb{Z}$.

4. Let $\mathcal{C}_c^\infty(\mathbb{R})$ be the space of differentiable functions on $\mathbb{R}$ with compact support, and let $L^1(\mathbb{R})$ be the completion of $\mathcal{C}_c^\infty(\mathbb{R})$ with respect to the L^1 norm. Let $f \in L^1(\mathbb{R})$. Prove that

$$\lim_{h\to 0} \frac{1}{h}\int_{|y-x|<h} |f(y) - f(x)| dy = 0$$

for almost every x.

Solution. Let X_k be the set of $x \in \mathbb{R}$ such that

$$\limsup_{h\to 0} \frac{1}{h}\int_{|y-x|<h} |f(y) - f(x)|\, dy > \frac{1}{k}.$$

We will show that X_k has measure 0 for each $k = 1, 2, \ldots$. The union of these sets is the set of x for which the displayed equation in the problem statement does not hold; if it is the union of countably many sets of measure 0, it also has measure 0, proving the desired statement.

Fix a positive integer k, and let $\varepsilon > 0$. By the given definition of $L^1(\mathbb{R})$, there is a differentiable function g on $\mathbb{R}$ with compact support such that $||f - g||_1 \le \varepsilon/4k$. Write $f_1 = f - g$. I claim that

$$\limsup_{h\to 0} \frac{1}{h}\int_{|y-x|<h} |f(y) - f(x)|\, dy = \limsup_{h\to 0} \frac{1}{h}\int_{|y-x|<h} |f_1(y) - f_1(x)|\, dy,$$

so we may replace f by f_1. Indeed, by the triangle inequality, the difference between the two sides is at most

$$\limsup_{h\to 0} \frac{1}{h}\int_{|y-x|<h} |g(y) - g(x)|\, dy.$$

Since g is continuous, we may choose h small enough so that the integrand is bounded by δ for any $\delta > 0$, hence this lim sup is 0.

So now suppose $f \in L^1(\mathbb{R})$ is such that $||f||_1 < \epsilon/4k$. Observe that

$$\begin{aligned}\limsup_{h\to 0} \frac{1}{h}\int_{|y-x|<h} |f(y)-f(x)|\,dy &\leq \limsup_{h\to 0} \frac{1}{h}\int_{|y-x|<h} |f(x)|+|f(y)|\,dy \\ &= 2\,|f(x)| + \limsup_{h\to 0} \frac{1}{h}\int_{|y-x|<h} |f(y)|\,dy.\end{aligned}$$

Now define $F(x) = \int_{-\infty}^{x} |f(y)|\,dy$. Then by the Lebesgue differentiation theorem F is differentiable with $F'(x) = |f(x)|$ for almost every x. The last term on the second line above equals $2F'(x)$ wherever the latter is defined, so for almost every x,

$$\limsup_{h\to 0} \frac{1}{h}\int_{|y-x|<h} |f(y)-f(x)|\,dy \leq 4\,|f(x)|.$$

The measure of the set of points x such that $4\,|f(x)| \geq 1/k$ is at most $4k\,||f||_1 < \varepsilon$, so the measure of X_k is at most ε. Since ε was arbitrary, X_k has measure 0 as claimed.

5. Let $\mathbb{P}^5$ be the projective space of homogeneous quadratic polynomials $F(X,Y,Z)$ over $\mathbb{C}$, and let $\Phi \subset \mathbb{P}^5$ be the set of those polynomials that are products of linear factors. Similarly, let $\mathbb{P}^9$ be the projective space of homogeneous cubic polynomials $F(X,Y,Z)$, and let $\Psi \subset \mathbb{P}^9$ be the set of those polynomials that are products of linear factors.

(a) Show that $\Phi \subset \mathbb{P}^5$ and $\Psi \subset \mathbb{P}^9$ are closed subvarieties.

(b) Find the dimensions of Φ and Ψ.

(c) Find the degrees of Φ and Ψ.

Solution.

(a) Identify $\mathbb{P}^2$ with the projective space of linear polynomials $F(X,Y,Z)$ over $\mathbb{C}$. Then there is a map $\mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^5$ given by multiplying the two linear polynomials to get a homogeneous quadratic polynomial. Its image is exactly Φ. Since $\mathbb{P}^2 \times \mathbb{P}^2$ is a projective variety, Φ is a closed subvariety of $\mathbb{P}^5$. Similarly, there is a map $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^9$ with image Ψ, showing that Ψ is a closed subvariety of $\mathbb{P}^9$.

(b) The fibers of the maps $\mathbb{P}^2 \times \mathbb{P}^2 \to \Phi$ and $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \to \Psi$ are all 0-dimensional by unique factorization, so $\dim \Phi = 4$ and $\dim \Psi = 6$.

(c) We will show that the degree of Ψ is 15. The degree of Φ can be shown to be 3 by a similar argument, or by noting that $\Phi \subset \mathbb{P}^5$ is defined by the vanishing of the determinant.

The dimension of Ψ is 6, so we could compute the degree of Ψ by intersecting Ψ with 6 generic hyperplanes in $\mathbb{P}^9$. Instead, we will choose 6 hyperplanes which are not generic. Each $f \in \Psi$ has a zero locus which is the union of three lines in $\mathbb{P}^2$. If x is a point of $\mathbb{P}^2$, the set of $g \in \mathbb{P}^9$ for which $g(x) = 0$ is a hyperplane. Pick 6 generic points $x_1, \ldots, x_6$ of $\mathbb{P}^2$, and consider those $f \in \Psi$ whose zero loci pass through all of these points. Such an f has a zero locus consisting of three lines whose union contains $x_1, \ldots, x_6$; there is exactly one way to choose those lines for each partition of $\{x_1, \ldots, x_6\}$ into three parts of size two. We can easily count that there are 15 such partitions. So Ψ meets this intersection of 6 hyperplanes set-theoretically in 15 points. Without verifying that the intersection is transverse, we can only conclude that the degree of Ψ is at least 15.

We next use the Hilbert polynomial to show that the degree of Ψ is at most 15. Let V_l be the vector space of degree-l homogeneous polynomials on Ψ, and W_l the vector space of degree-(l, l, l) tri-homogeneous polyonimals on $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ which are invariant under the action of S_3 given by permuting the three $\mathbb{P}^2$ factors. Name the multiplication map $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^9$ from part (a) m. Pullback along m gives a map m^* from V_l to W_l, because $m \circ \sigma = m$ for any $\sigma \in S_3$ acting on $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$. Moreover m^* is injective, since m is surjective. Therefore $\dim V_l \leq \dim W_l$. The dimension of W_l is the number of monomials of tridegree (l, l, l) up to symmetry, or equivalently the number of 3×3 matrices of nonnegative integers with columns summing to l up to permutation of columns. There are $\binom{l+2}{2}$ possible columns and thus $\binom{\binom{l+2}{2}+2}{3} = \frac{l^6}{2^3 \cdot 6} + O(l^3)$ such matrices. So $\dim V_l \leq \frac{l^6}{2^3 \cdot 6} + O(l^3)$ and it follows that the degree of Ψ is at most $\frac{6!}{2^3 \cdot 6} = 15$. Together with the previous bound, this shows that $\deg \Psi = 15$.

(Note: m^* is not always surjective. The dimension of V_l is at most the dimension of the space of degree-l homogeneous polynomials on $\mathbb{P}^9$, namely $\binom{9+l}{l}$. When $l = 2$ this is only $\binom{11}{2} = 55$, while $\dim W_l = \binom{\binom{4}{2}+2}{3} = \binom{8}{3} = 56$. Thus, additional care would be needed to show that $\deg \Psi = 15$ using only the Hilbert polynomial.)

6. Realize S^1 as the quotient $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, and consider the following two line bundles over S^1:

L is the subbundle of $S^1 \times \mathbb{R}^2$ given by

$$L = \{(\theta, (x, y)) : \cos(\theta) \cdot x + \sin(\theta) \cdot y = 0\}; \text{and}$$

M is the subbundle of $S^1 \times \mathbb{R}^2$ given by

$$M = \{(\theta, (x, y)) : \cos(\theta/2) \cdot x + \sin(\theta/2) \cdot y = 0\}.$$

(You should verify for yourself that M is well-defined.) Which of the following are trivial as vector bundles on S^1?

(a) L

(b) M

(c) $L \oplus M$

(d) $M \oplus M$

(e) $M \otimes M$

Solution.

(a) Since L is a line bundle, to show that L is trivial, it suffices to give a section of L which is everywhere nonzero. Take $s(\theta) = (-\sin(\theta), \cos(\theta))$.

(b) Let $B \subset M$ be the subbundle of vectors of unit length (so B is an S^0 bundle over S^1). Consider the map $\gamma : S^1 = \mathbb{R}/2\pi\mathbb{Z} \to B$ defined by $\gamma(\theta) = (2\theta, (-\sin(\theta), \cos(\theta)))$. Then γ is a homeomorphism, so in particular, B is not homeomorphic to $S^0 \times S^1$, and M cannot be a trivial line bundle.

(c) Let $C \subset L \oplus M$ be the subbundle of vectors of unit length (so C is an S^1 bundle over S^1). We will write $v \oplus w$ for a vector in $L \oplus M$ over $x \in S^1$, where v and w are vectors in L and M over x respectively. Consider the map $h : S^1 \times [0, 2\pi] \to C$ given by

$$h(\phi, \theta) = (\theta, (\cos\phi(-\sin\theta, \cos\theta) \oplus \sin\phi(-\sin(\theta/2), \cos(\theta/2)))).$$

This is a homotopy between the maps $S^1 \to C$ given by

$$h(\phi, 0) = (0, ((0, \cos\phi) \oplus (0, \sin\phi)))$$

and

$$h(\phi, 2\pi) = (0, ((0, \cos\phi) \oplus (0, -\sin\phi))).$$

If $L \oplus M \to S^1$ were a trivial plane bundle, then C would be the torus and these two paths would not be homotopic. Hence $L \oplus M$ is not a trivial plane bundle over S^1.

(d) Define $s : [0, 2\pi] \to M \oplus M$ by

$$s(\theta) = (\theta, (\cos(\theta/2)(-\sin(\theta/2), \cos(\theta/2)) \oplus \sin(\theta/2)(-\sin(\theta/2), \cos(\theta/2)))).$$

Observe that s is nowhere 0 and $s(0) = (0, ((0, 1) \oplus (0, 0)))$ is equal to $s(2\pi) = (0, (-(0, -1) \oplus (0, 0)))$. So s factors through S^1, and thus is a global nonvanishing section of $M \oplus M$. We can get a second, linearly independent section of $M \oplus M$ by applying the map $A : M \oplus M \to M \oplus M$,

$$A(\theta, (v \oplus w)) = (\theta, ((-w) \oplus v))$$

to s. So s and $A \circ s$ form a basis for $M \oplus M$ at every point, and $M \oplus M$ is a trivial plane bundle over S^1.

(e) Consider the map $s : [0, 2\pi] \to M$ given by

$$s(\theta) = (\theta, (-\sin(\theta/2), \cos(\theta/2))).$$

Since $s(0) = (0, (0, 1))$ while $s(2\pi) = (0, (0, -1))$, s does not factor through S^1. However, if we define $s' : [0, 2\pi] \to M \otimes M$ by

$$s'(\theta) = (\theta, v \otimes v) \quad \text{where} \quad (\theta, v) = s(\theta),$$

then $s'(0) = (0, (0, 1) \otimes (0, 1)) = (0, (0, -1) \otimes (0, -1)) = s'(2\pi)$. So s' is a global nonvanishing section of the line bundle $M \otimes M$, and thus $M \otimes M$ is trivial.

Note: Parts (c)–(e) can be solved more systematically using the theory of vector bundles. For X a pointed compact space, an n-dimensional vector bundle on the suspension of X is determined up to isomorphism by a homotopy class of pointed maps from X to the orthogonal group $O(n)$. For a map $f : X \to O(n)$, the corresponding vector bundle is obtained by taking trivial bundles on two copies of the cone on X and identifying them at a point $x \in X$ via the map $f(x)$. In our case $X = S^0$ and so a homotopy class of pointed maps from X to $O(n)$ is just a connected component of $O(n)$. The bundles L and M correspond to the connected components of the matrices (1) and (-1) respectively. It follows that the bundles $L \oplus M$, $M \oplus M$, and $M \otimes M$ correspond to

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 \end{pmatrix},$$

respectively, so $L \oplus M$ is nontrivial but $M \oplus M$ and $M \otimes M$ are trivial.