

Qualifying exam, Spring 2007, Day 1

All problems are worth 10 points.

Problem 1.

- (a) Let T be a linear map $V \rightarrow V$, where V is a finite-dimensional vector space over an algebraically closed k . Show that V admits a non-zero eigenvector.
- (b) For T as above, show that V can be decomposed as a direct sum $V \simeq \bigoplus_i V_i$ of T -stable subspaces V_i , such that for every i the restriction of T to V_i equals the sum of a scalar operator T'_i , and a *nilpotent* operator T''_i such that $(T''_i)^{\dim(V_i)-1} \neq 0$.
- (c) Deduce that an $n \times n$ matrix over k can be conjugated to one in the Jordan canonical form.

Problem 2. Let P_1, P_2, P_3, P_4 be a collection of 4 distinct points in \mathbb{P}^2 . Show that this set is a complete intersection if and only if either all 4 points lie on the same line, or no 3 points lie on the same line.

Problem 3. Let $f \in \mathbb{C}(z)$ be a rational function. Assume that f has no poles on the closed right half-plane $\operatorname{Re}(z) \geq 0$ and is bounded on the imaginary axis. Show that f is bounded on the right half-plane.

Problem 4. Show that S^3 minus two linked circles is homotopy equivalent to $S^1 \times S^1$.

Problem 5.

- (a) Define orientability of a smooth manifold.
- (b) Let X be a n -dimensional submanifold of \mathbb{R}^{n+1} . Show that X is orientable if and only if there exists a nowhere-vanishing normal vector field along X .

Problem 6. Let V be a separable Banach space, and let V^* be its continuous dual. Consider the closed unit ball $B(0, 1) \subset V^*$ endowed with the *weak* topology. Construct a norm $\|\cdot\|_{new}$ on V^* such that the above topology is equivalent to one corresponding to this norm.

Qualifying exam, Spring 2007, Day 2

All problems are worth 10 points.

Problem 1.

- (a) Show that $\mathbb{Q}(\sqrt{2} + \sqrt{3}) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$.
- (b) Let M/K be a Galois field extension, and let $K \subset L_1, L_2 \subset M$ be two intermediate fields. Show that $L_1 \simeq L_2$ as K -algebras if and only if $\text{Gal}(M/L_1)$ is conjugate to $\text{Gal}(M/L_2)$ as subgroups of $\text{Gal}(M/K)$.

Problem 2. Construct an isomorphism from a smooth conic in \mathbb{CP}^2 to \mathbb{CP}^1 .

Problem 3.

- (a) Prove the *Schwarz Lemma*: Let $f(z)$ be analytic for $|z| < 1$ and satisfy the conditions $f(0) = 0$ and $|f(z)| \leq 1$; then $|f(z)| \leq |z|$.
- (b) Let $f(z)$ be analytic on the closed unit disk and satisfy $|f(z)| \leq c < 1$. Show that there is a unique solution in the unit disk to the equation $f(z) = z$.

Problem 4. Consider the topological space, obtained from an annulus by identifying antipodal points on the outer circle and antipodal points on the inner circle (separately). Compute its fundamental group.

Problem 5. Let S be a compact surface contained in a closed ball in \mathbb{R}^3 of radius r . Show that there exists at least one point $p \in S$ where the Gauss curvature and the absolute value of mean curvature are bounded below by $\frac{1}{r^2}$ and $\frac{1}{r}$ respectively.

Problem 6. Let $A = [a_{ij}]$ be a real symmetric $n \times n$ matrix. Define $f : \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$f(x_1, \dots, x_n) = \exp\left(-\sum a_{ij}x_i x_j\right).$$

Prove that $f \in L^1(\mathbb{R}^n)$ if and only if the matrix A is positive definite. Compute $\|f\|_1$ if this is the case.

Qualifying exam, Spring 2007, Day 3

*All problems are worth 10 points. Problems marked with * give extra bonus.*

Problem 1. Let G be a group and M a G -module (i.e., an abelian group, endowed with an action of G).

- (a) Define the cohomology group $H^1(G, M)$.
 (b) Show that $H^1(G, M)$ is in bijection with the set of isomorphism classes of short exact sequences of G -modules

$$0 \rightarrow M \rightarrow \tilde{M} \rightarrow \mathbb{Z} \rightarrow 0,$$

where \mathbb{Z} is endowed with the trivial G -action. (We say that two such short exact sequences are isomorphic if there exists an isomorphism of G -modules $\tilde{M}_1 \rightarrow \tilde{M}_2$, which maps $M \subset \tilde{M}_1$ *identically* to $M \subset \tilde{M}_2$, and such that induced map $\mathbb{Z} \simeq \tilde{M}_1/M \rightarrow \tilde{M}_2/M \simeq \mathbb{Z}$ is also the identity map.)

Problem 2. Let $X \subset \mathbb{A}^{2n^2}$ be the variety of pairs (A, B) , where A and B are $n \times n$ matrices such that $A \cdot B = 0$ and $B \cdot A = 0$.

- (a) Show that each irreducible component of X has dimension n^2 .
 (b*) Show that each irreducible component of X is smooth away from the locus where it meets other irreducible components.

Problem 3. Prove that the cosecant function can be written as

$$\csc z = \frac{1}{z} - 2z \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 \pi^2 - z^2},$$

for $z \in \mathbb{C}$, $z \neq k\pi$ ($k \in \mathbb{Z}$).

Problem 4. Calculate the cohomology $H^i(X, \mathbb{Z})$ (just as abelian groups), where X is the Grassmannian of 2-planes in \mathbb{C}^4 .

Problem 5.

- (a) Let G, H be Lie groups. Let $\phi : G \rightarrow H$ be a homomorphism and let K be the kernel of ϕ . Assume that G is connected and K is discrete. Show that K lies in the center of G .
 (b) Deduce that the fundamental group of a Lie group is abelian.

Problem 6. Prove Picard's theorem: Let v be a continuously differentiable time-dependent vector field defined on a domain of $\mathbb{R} \times \mathbb{R}^n$, containing a point (t_0, x_0) . Show that there exists an open interval $t_0 \in I \subset \mathbb{R}$ and a unique differentiable function $\gamma : I \rightarrow \mathbb{R}^n$ with $\gamma(t_0) = x_0$ and $d\gamma|_t = v(t, \gamma(t))$.