Math 99r Tutorial topics

(2015-2016)

Fall Tutorial 2015

Category Theory

Description: Category theory is central to the study of modern mathematics. Most of the subjects which have been profoundly influenced are, however, rather technical and typically lie outside of the standard undergraduate curriculum. On the other hand, insights gained from a categorical point of view can be very useful when learning mathematics at any level. We will introduce the subject by building on many familiar examples from basic algebra and topology. We will explain the meaning of categories, functors, natural transformations, limits, colimits, Yoneda's lemma, representability, and adjunctions. In order to make our exposition more meaningful and enjoyable, each definition will be followed by an explicit example in a category that the students are acquainted with. Once the basis of the subject has been throughly developed, we will venture into more advanced topics. Potential topics may include, but are not limited to:

(1) Abelian categories and homological algebra,
(2) Model categories,
(3) Higher categories.

Students' backgrounds and interests will be the main parameters taken into consideration when choosing topics.

Prerequisites: Math 122 and Math 131.

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Spring Tutorial 2016

Morse Theory

Description: Morse theory enables one to study the topology of smooth manifolds by studying smooth functions. According to Marston Morse, a "generic" smooth function can reflect the topology of the underlying manifold quite directly.

By definition, a Morse function on a smooth manifold is a smooth function whose Hessians are non-degenerate at critical points. One can proof that any function can be perturbed to a Morse function. The classical approach of Morse theory is to construct a

cellular decomposition of the manifold from the Morse function and then study the topology of the manifold accordingly. Each critical point of the Morse function corresponds to a cell with dimension equal to the number of negative eigenvalues of the Hessian matrix.

However, the classical method cannot be generalized to infinite dimensional cases because attaching a cell of infinite dimension does not change the homotopy type. An approach for the infinite dimensional case was introduced by Andreas Floer: instead of studying the cellular decomposition, he considered the flow lines of the Morse function and defined homology groups out of it. If the dimension of the manifold is finite, such groups are canonically isomorphic to the singular homology groups. In the infinite dimensional case, this turns out to be a new invariant and is now called "Floer homology".

In the tutorial, we are going to introduce both the classical and Floer's approach of Morse theory on finite dimensional manifolds. Then, as an application, we will talk about the smooth h-cobodism theorem and give a proof of the generalized Poincare conjecture in higher dimensions. If time allows, we will talk about other applications of Morse theory such as the Lefschetz hyperplane theorem and the existence of closed geodesics on compact Riemannian manifolds.

Prerequisites: Students need to be familiar with the concepts of smooth manifolds, tangent and cotangent bundles, and tangent maps. Some knowledge in homology theory would be helpful but is not required.

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Partitions, Young Diagrams and Beyond

Quotation: The theory of partitions is one of the very few branches of mathematics that can be appreciated by anyone who is endowed with little more than a lively interest in the subject. Its applications are found wherever discrete objects are to be counted or classified, whether in the molecular and the atomic studies of matter, in the theory of numbers, or in combinatorial problems from all sources. Gian-Carlo Rota

Description: A partition of an integer n is a finite weakly decreasing sequence of positive integers with a sum equal to n. It can be visualizedas a Young diagram: a collection of cells arranged in left-justified rows with row lengths given by elements of the sequence.

Despite such elementary description, Young diagrams occur in a variety of interplays between combinatorial and algebraic structures, related in particular to group representation theory, algebra of symmetric polynomials, and beautiful identities with series. They lead to surprising connections, in particular the knowledge about Young diagrams and representations of the symmetric group sheds light on questions such as:

How many times do you need to shuffle a deck of cards to make it close to random? Or: What can we say about the length of a longest increasing subsequence of a random permutation? Such interplays and their consequences will be the focus of this tutorial.

Prerequisites: Only basic linear algebra and interest in combinatorics. All other notions will be introduced during the course.

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