The Maximal Rank Conjecture for Sections of Curves

Eric Larson

Abstract

Let $C \subset \mathbb{P}^r$ be a general curve of genus g embedded via a general linear series of degree d. The well-known Maximal Rank Conjecture asserts that the restriction maps $H^0(\mathcal{O}_{\mathbb{P}^r}(m)) \to H^0(\mathcal{O}_C(m))$ are of maximal rank; if known, this conjecture would determine the Hilbert function of C.

In this paper, we prove an analogous statement for the hyperplane sections of general curves. More specifically, if $H \subset \mathbb{P}^r$ is a general hyperplane, we show that $H^0(\mathcal{O}_H(m)) \to H^0(\mathcal{O}_{C\cap H}(m))$ is of maximal rank, except for some counterexamples when $m = 2$. We also prove a similar theorem for the intersection of a general space curve with a quadric.

1 Introduction

Let $\mathcal{H}_{d,g,r}$ denote the Hilbert scheme classifying subschemes of $\mathbb{P}^r$ with Hilbert polynomial $P(x) = dx + 1 - g$. We have a natural rational map from any component of $\mathcal{H}_{d,g,r}$ whose general member is a smooth curve to the moduli space M_g of curves. The Brill-Noether theorem asserts that there exists such a component that dominates M_g if and only if

$$\rho(d, g, r) := (r + 1)d - rg - r(r + 1) \geq 0.$$

Moreover, it is known that when $\rho(d, g, r) \geq 0$, there exists a unique such component that dominates M_g. We shall refer to a curve $C \subset \mathbb{P}^r$ lying in this component as a *Brill-Noether Curve (BN-curve)*.

A natural first step in understanding the extrinsic geometry of general curves is to understand their Hilbert function. Here, we have the celebrated *Maximal Rank Conjecture*:

Conjecture 1.1 (Maximal Rank Conjecture)**.** If C is a general BN-curve and m is a positive integer, then the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^r}(m)) \to H^0(\mathcal{O}_C(m))$$

is of maximal rank.

Remark 1.2. Since $H^1(\mathcal{O}_C(m)) = 0$ for $m \geq 2$ when C is a general BN-curve, the maximal rank conjecture would completely determine the Hilbert function of C.

In this paper, we prove that the general hyperplane section of a general BN-curve imposes the expected number of conditions on hypersurfaces of every degree, apart from a few counterexamples that occur for quadric hypersurfaces. More precisely, we prove:

Theorem 1.3 (Hyperplane Maximal Rank Theorem)**.** *If C is a general BN-curve, $H \subset \mathbb{P}^r$ is a general hyperplane, and m is a positive integer, then the restriction map*

$$H^0(\mathcal{O}_H(m)) \to H^0(\mathcal{O}_{C\cap H}(m))$$

is of maximal rank, except possibly when $m = 2$ and $d < g + r$.

The conclusion that this restriction map is of maximal rank can be reformulated in terms of the cohomology of the twists of the ideal sheaf as follows:

$$H^0(\mathcal{I}_{(C\cap H)/H}(m)) = 0 \quad \text{when } d \geq \binom{m+r-1}{r-1},$$
$$H^1(\mathcal{I}_{(C\cap H)/H}(m)) = 0 \quad \text{when } d \leq \binom{m+r-1}{r-1}.$$

As a further application of the techniques developed, we also prove an analogous statement for quadric sections of space curves. More precisely:

Theorem 1.4. *If $C \subset \mathbb{P}^3$ is a general BN-curve, $Q \subset \mathbb{P}^r$ is a general (smooth) quadric hypersurface, and (m, n) are nonnegative integers, then the restriction map*

$$H^0(\mathcal{O}_Q(m,n)) \to H^0(\mathcal{O}_{C\cap Q}(m,n))$$

is of maximal rank, unless we are in one of the following cases:

$\boldsymbol{(m,n)}$	$\boldsymbol{(d,g)}$
$(2,2)$	$(6,4)$, $(5,2)$, *or* $(4,1)$
$(3,3)$	$(6,4)$, $(8,6)$, *or* $(7,5)$
$(2,3)$	$(6,4)$.

We shall prove these theorems using an inductive approach due originally to Hirschowitz [2]. In its simplest form, suppose that $C = X \cup Y$ is a reducible curve such that Y is contained in some hyperplane H'. Then we have the exact sequence of sheaves

$$0 \to \mathcal{I}_{(X\cap H)/H}(m-1) \to \mathcal{I}_{(C\cap H)/H}(m) \to \mathcal{I}_{(Y\cap H)/(H\cap H')}(m) \to 0,$$

which gives rise to a long exact sequence in cohomology

$$\cdots \to H^i(\mathcal{I}_{(X\cap H)/H}(m-1)) \to H^i(\mathcal{I}_{(C\cap H)/H}(m)) \to H^i(\mathcal{I}_{(Y\cap H)/(H\cap H')}(m)) \to \cdots.$$

Consequently, we can deduce the hyperplane maximal rank theorem for the general hyperplane section of C from the hyperplane maximal rank theorem for the general hyperplane sections of X and Y.

Traditionally, this method has been applied only when the curve Y is *nonspecial*, i.e. satisfies $H^1(\mathcal{O}_Y(1)) = 0$; one main difference between this paper and previous work related to the maximal rank conjecture is the use of this inductive method for special curves Y, which requires a more delicate analysis.

The structure of this paper is as follows. First, in Section 2, we give several methods of constructing reducible BN-curves that will be useful for specialization arguments later on.

In Sections 3 and 4, we prove the hyperplane maximal rank theorem in the special cases $r = 3$ and $m = 2$ respectively. We then deduce the general case in Sections 5 and 6 via the above inductive argument, by finding appropriate BN-curves $X \subset \mathbb{P}^r$ and $Y \subset H' \subset \mathbb{P}^r$ satisfying the hyperplane maximal rank theorem for $(m-1, r)$ and $(m, r-1)$ respectively. Finally, in Section 7, we apply the techniques developed in Section 3 to prove Theorem 1.4.

Notational Convention: We say a BN-curve $X \subset \mathbb{P}^r$ is *nonspecial* if $d \geq g + r$, i.e. if X is a *limit* of curves with nonspecial hyperplane section.

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2 Some Gluing Lemmas

In this section, we will give some lemmas that let us construct examples of BN-curves.

Lemma 2.1. *Let $X \subset \mathbb{P}^r$ be a curve with $H^1(N_X) = 0$, and D be a rational normal curve of degree $d \leq r$ that is k-secant to X, where*

$$k \leq \begin{cases} d+1 & \text{if } d < r; \\ r+2 & \text{if } d = r. \end{cases}$$

Then $X \cup D$ is smoothable and $H^1(N_{X \cup D}) = 0$. Moreover, if X is a BN-curve, then $X \cup D$ is a BN-curve.

Proof. The vanishing of $H^1(N_{X \cup D})$ and smoothability of $X \cup D$ are consequences of Theorem 4.1 of [1] (via the same argument as Corollary 4.2 of [1]), together with the fact that

$$N_D = \mathcal{O}_{\mathbb{P}^1}(d)^{\oplus(r-d)} \oplus \mathcal{O}_{\mathbb{P}^1}(d+2)^{\oplus(d-1)}.$$

Now assume X is a BN-curve. To show that $X \cup D$ is a BN-curve, we just need to count the dimension of the space of embeddings of $X \cup D$ into projective space (this suffices because there is a unique component of the Hilbert scheme that dominates M_g). In order to do this, first note that

$$\rho(X \cup D) = \rho(X) + (r+1)d - r(k-1).$$

Consequently, the verification that $X \cup D$ is a BN-curve boils down to the following two assertions, both of which are straight-forward to check:

1. Given a $\mathbb{P}^1$ with $k \leq d+1$ marked points, the family of degree d embeddings of $\mathbb{P}^1$ as a rational normal curve with given values at the marked points has dimension

$$(r-d)(d-k+1) + d(d+2-k) = (r+1)d - r(k-1).$$

2. Given a $\mathbb{P}^1$ with $r+2$ marked points, there is a unique embedding of $\mathbb{P}^1$ as a rational normal curve of degree r with given values at all marked points.

This completes the proof. $\square$

Lemma 2.2. *Let $X \subset \mathbb{P}^r$ be a curve with $H^1(N_X) = 0$, and R be a rational normal curve of degree $r-1$ that is $(r+1)$-secant to X, and L be a line that is 1-secant to both X and R. Then $H^1(N_{X\cup R\cup L}) = 0$.*

Proof. Note that for curves A and B,

$$H^1(N_{A\cup B}|_A) = 0 \quad \text{and} \quad H^1(N_{A\cup B}|_B(-A\cap B)) = 0 \quad \Rightarrow \quad H^1(N_{A\cup B}) = 0;$$

indeed, this holds for $N_{A\cup B}$ replaced by any vector bundle.

In particular, since N_A is a subbundle of full rank in $N_{A\cup B}|_A$, we can conclude that $H^1(N_{A\cup B}) = 0$ provided that

$$\begin{aligned} H^1(N_A) = 0 \quad &\text{and} \quad H^1(N_{A\cup B}|_B(-A\cap B)) = 0, \\ \text{or respectively} \quad H^1(N_{A\cup B}|_A) = 0 \quad &\text{and} \quad H^1(N_B(-A\cap B)) = 0. \end{aligned}$$

Thus, the vanishing of $H^1(N_{X\cup R\cup L})$ follows from the following facts:

$$\begin{aligned} H^1(N_X) &= 0 \\ H^1(N_{R\cup L}|_R(-X\cap R)) &= H^1(\mathcal{O}_{\mathbb{P}^1}^{\oplus(r-2)} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) = 0. \\ H_1(N_L(-L\cap(X\cup R))) &= H^1(\mathcal{O}_{\mathbb{P}^1}(-1)) = 0. \end{aligned}$$

$\square$

Lemma 2.3. *Let $X \subset \mathbb{P}^r$ be a curve with $H^1(N_X) = 0$, and L be a line 3-secant to X. Assume that the tangent lines to X at the three points of intersection do not all lie in a plane. Then $X \cup D$ is smoothable and $H^1(N_{X\cup D}) = 0$.*

Proof. See Remark 4.2.2 of [1]. $\square$

We end this section with a simple observation, that will be used several times in the remainder of the paper and will therefore be useful to spell out.

Lemma 2.4. *Let $\mathcal{X}$ and $\mathcal{Y}$ be irreducible families of curves in $\mathbb{P}^r$, sweeping out subvarieties $\overline{\mathcal{X}}, \overline{\mathcal{Y}} \subset \mathbb{P}^r$ of codimension at most one. Let X and Y be specializations of $\mathcal{X}$ and $\mathcal{Y}$ respectively, such that $X \cup Y$ is a BN-curve with $H^1(N_{X\cup Y}) = 0$, and $X \cap Y$ is quasi-transverse and general in $\overline{\mathcal{X}} \cap \overline{\mathcal{Y}}$.*

Then there are simultaneous generalizations X' and Y' of X and Y respectively such that $X' \cup Y'$ is a BN-curve with $\#(X\cap Y) = \#(X'\cap Y')$. Equivalently, in more precise language, write B_1 and B_2 for the bases of $\mathcal{X}$ and $\mathcal{Y}$ respectively. Then we are asserting the existence of an irreducible $B \subset B_1 \times B_2$ dominating both B_1 and B_2, such that any fiber (X', Y') of $(\mathcal{X} \times \mathcal{Y}) \times_{(B_1\times B_2)} B$ satisfies the given conclusion.

Proof. As $\overline{\mathcal{Y}}$ has codimension at most one, the intersection of any generalization X' of X with $\overline{\mathcal{X}} \cap \overline{\mathcal{Y}}$ contains a generalization of $X \cap Y$. Similarly, the intersection of any generalization Y' of Y with $\overline{\mathcal{X}} \cap \overline{\mathcal{Y}}$ contains a generalization of $X \cap Y$. The existence of simultaneous generalizations X' and Y' of X and Y respectively with $\#(X\cap Y) = \#(X'\cap Y')$ thus follows from the generality of $X \cap Y$ in $\overline{\mathcal{X}} \cap \overline{\mathcal{Y}}$.

Moreover, since $H^1(N_{X\cup Y}) = 0$, the curve $X \cup Y$ is a smooth point of the corresponding Hilbert scheme; consequently, any generalization $X' \cup Y'$ of $X \cup Y$ is a BN-curve. $\square$

3 The Case $r = 3$

In this section, we will prove that if $C \subset \mathbb{P}^3$ is a general BN-curve, then

$$H^0(\mathcal{O}_H(m)) \to H^0(\mathcal{O}_{C\cap H}(m))$$

is of maximal rank, unless C is a canonically embedded curve of genus 4 and $m = 2$. (In which case by inspection the above map fails to be of maximal rank.) We will do this by specializing C to a particular family of curves and computing the plane sections of these curves.

Definition 3.1. A *defining curve* is a curve of the form $C \cup \{L_i\} \cup \{M_i\} \cup \{R_i\}$, where:

1. C is a rational normal curve.
2. The L_i are general 1-secant lines to C.
3. The M_i are general 2-secant lines to C.
4. The R_i are general 5-secant rational normal curves of degree 3 to C.

We call $(\#\{L_i\}, \#\{M_i\}, \#\{R_i\})$ the *signature* of the defining curve.

Note that defining curves are BN-curves by Lemma 2.1. If X is a defining curve with signature (a, b, c), then its degree is $a + b + 3(c + 1)$, and its genus is $b + 4c$. In particular, for any (d, g) with $\rho(d, g, 3) \geq 0$, there is a defining curve of degree d and genus g in $\mathbb{P}^3$.

The first natural question here is thus to understand what families of 6 points in the plane can be realized as the plane section of two rational normal curves meeting at 5 points.

Lemma 3.2. *Fix* 6 *general points* $q_1, q_2, q_3, p_1, p_2, p_3 \in \mathbb{P}^2 \subset \mathbb{P}^3$ *lying on a smooth conic, and let* C *be a rational normal curve through* $\{q_1, q_2, q_3\}$*. Then there a rational normal curve* D *containing* $\{p_1, p_2, p_3\}$ *such that* $\#(C \cap D) = 5$*.*

Proof. Pick a smooth quadric $Q \subset \mathbb{P}^3$ that contains these 6 points and the curve C (this is possible as C is contained in a 3-dimensional family of quadrics, it is two conditions for a conic to pass though $\{p_1, p_2\}$, and once a conic passes through $\{q_1, q_2, q_3, p_1, p_2\}$, it must also pass through p_3).

As C is a rational normal curve, C is a curve of type, without loss of generality, $(1, 2)$ on $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Now let D be a smooth curve of type $(2, 1)$ on Q passing through $\{p_1, p_2, p_3\}$. As D is a curve of type $(2, 1)$ on Q, it is a rational normal curve. Moreover, by intersection theory on Q, we have $\#(C \cap D) = 5$. This completes the proof. $\square$

Remark 3.3. Conversely, if such a curve D exists, then the six points $q_1, q_2, q_3, p_1, p_2, p_3$ must lie on a conic. Indeed, any two rational normal curves meeting at 5 points must lie on a quadric; restricting that quadric to the plane implies that $q_1, q_2, q_3, p_1, p_2, p_3$ must lie on a conic.

This prompts the following definition:

Definition 3.4. A *conic collection* of points is a set of $3c + 3$ points $q_1, q_2, q_3, p_1, p_2, \ldots, p_{3c}$ in the plane such that $\{q_1, q_2, q_3, p_{3k+1}, p_{3k+2}, p_{3k+3}\}$ lie on a conic for $0 \leq k \leq c - 1$.

Corollary 3.5. *Let $S \subset \mathbb{P}^2$ be a general set of $a+b+3c+3$ points, subject to the restriction that some subset with cardinality $3c+3$ of S is a conic collection of points. Then there is a defining curve of signature (a,b,c) whose intersection with $\mathbb{P}^2$ is S.*

Proof. This follows from Lemma 3.2, plus the observation that for a rational normal curve C and general point $p \in \mathbb{P}^2$, we can find a 1-secant (respectively, 2-secant) line to R passing through p. $\square$

In light of this, it suffices to prove that the sets of points S appearing in Corollary 3.5 impose independent conditions on polynomials of degree m, unless we are in the case corresponding to a canonical curve of genus 4, i.e.

$$(a,b,c) = (0,0,1) \quad \text{and} \quad m = 2.$$

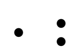
To do this, we will use a method similar to Hirschowitz's method outlined in the introduction, with the role of the hyperplane being played by a plane conic. In order to do this, we will need to figure out how to appropriately specialize the sets S appearing in Corollary 3.5.

Definition 3.6. Let b and c be nonnegative integers. Start with j empty columns, and consider the following game, where we perform the first step b times, and our choice of the remaining steps c times.

1. Pick any column and add a dot to it.

2. Pick any three columns and place a dot in each one.

3. Pick any two columns and place a single dot in the first one and two dots in the second one.



4. Pick any column and add three dots to it.

We say a sequence of nonnegative integers $(n_1, n_2, \ldots, n_j)$ is (b,c)-*reachable* if we can do this so there are n_k dots in the kth column.

Lemma 3.7. *Let b and c be integers, and $Q_1, Q_2, \ldots, Q_j$ be general conics passing through $\{q_1, q_2, q_3\}$. If $S = \{p_1, p_2, \ldots, p_{b+3c}\} \subset \mathbb{P}^2$ is a general set of $b+3c$ points subject to the constraint that $\{q_1, q_2, q_3, p_1, p_2, \ldots, p_{3c}\}$ forms a conic collection of points, and $(n_1, n_2, \ldots, n_j)$ is (b,c)-reachable, then we can specialize S to a subscheme $S^0 \subset \mathbb{P}^2$ with*

$$\deg\left(S^0 \cap Q_k \smallsetminus (Q_1 \cup Q_2 \cup \cdots \cup Q_{k-1})\right) = n_k.$$

Proof. We use induction on c; the base case $c = 0$ is obvious. When we increase c by one, we add 3 points $\{p_2, p_1, p_0\}$ lying on a general quadric Q passing through $\{q_1, q_2, q_3\}$, and we add three dots to the columns (i_2, i_1, i_0), respectively (i_2, i_0), respectively (i_0). In the first case, we begin by specializing p_1 and p_2 to the remaining points of intersection of Q with Q_{i_1} and Q_{i_2} respectively; similarly, in the second case, we begin by specializing p_2 to the remaining point of intersection of Q with Q_{i_2}. After this, we specialize Q to Q_{i_0} while preserving these incidence relations. $\square$

Lemma 3.8. *A sequence $(n_1, n_2, \ldots, n_j)$ with $\sum n_i = b + 3c$ is (b, c)-reachable if and only if neither of the following hold:*

1. *$b = 0$, and $j = 2$, and $n_2 = 1$.*
2. *$b = 0$, and $j = 3$, and $n_2 = 2$, and $n_3 = 1$.*

Proof. We will induct on c; the base case $c = 0$ is obvious. For the inductive step, we will proceed by induction on b. First we suppose that $b = 0$. For a sequence $(m_1, m_2, \ldots, m_k)$, write $(m_1, m_2, \ldots, m_k)^0$ for the same sequence with zeros removed. We consider two cases.

Case 1: $\boldsymbol{n_j \neq 1}$. If $j = 1$, the statement is clear, so we assume $j \geq 2$. If $n_1 \geq 3$, then applying the inductive hypothesis to $(n_1 - 3, n_2)^0$ with $(0, b - 1)$ gives the desired result, so we assume also that $n_1 \leq 2$.

If $j = 2$, then we apply the inductive hypothesis to $(n_1 - 1, n_2 - 2)^0$ with $(0, c - 1)$; this gives the desired result because if $n_1 = 2$, then $n_2 \geq 4$.

If $j = 3$ and $n_2 \geq 2$, then we apply the inductive hypothesis to $(n_1 - 1, n_2 - 2, n_3)^0$ with $(0, c-1)$. For $j = 3$ and $n_2 = 1$, we apply the inductive hypothesis to $(n_1 - 1, n_2 - 1, n_3 - 1)^0$ with $(0, c - 1)$; this gives the desired result because if $(n_1, n_2) = (2, 1)$, then $n_3 \geq 3$.

Finally, for $j \geq 4$, we apply the inductive hypothesis to $(n_1-1, n_2-1, n_3-1, n_4, n_5, \ldots, n_j)^0$ with $(0, c - 1)$.

Case 2: $\boldsymbol{n_j = 1}$. Our assumptions imply $j \geq 3$. If $j = 3$, then we apply the inductive hypothesis to $(n_1 - 1, n_2 - 1, n_3 - 1)^0$ with $(0, c - 1)$; this gives the desired result because $n_2 \neq 2$ by assumption. Similarly, if $j \geq 4$ and $n_{j-1} \neq 1$, then we apply the inductive hypothesis to $(n_1 - 1, n_2 - 1, n_3, n_4, \ldots, n_{j-1}, n_j - 1)^0$ with $(0, c - 1)$. Finally, if $j \geq 4$ and $n_{j-1} = 1$, then we apply the inductive hypothesis to $(n_1, n_2, \ldots, n_{j-2})$ with $(1, c - 1)$.

This completes the proof of the base case $b = 0$; next, we consider $b = 1$. If $n_j = 1$ and $n_{j-1} \neq 1$, then we apply the inductive hypothesis to $(n_1, n_2, \ldots, n_{j-1}, n_j - 1)^0$ with $(0, c)$; otherwise, we apply the inductive hypothesis to $(n_1 - 1, n_2, n_3, \ldots, n_j)^0$ with $(0, c)$.

For $b \geq 2$, we apply the inductive hypothesis to $(n_1 - 1, n_2, n_3, \ldots, n_j)^0$ with $(0, c)$. $\square$

Lemma 3.9. *Let $(n'_1, n'_2, \ldots, n'_j)$ be a sequence, and b and c be nonnegative integers. Unless $\sum n'_k = b + 3c$ and $(n'_1, n'_2, \ldots, n'_j)$ is not (b, c)-reachable, there is a (b, c)-reachable sequence $(n_1, n_2, \ldots, n_j)$ for which*

$$n_k \geq n'_k \qquad \text{if} \quad \sum n'_k \geq b + 3c;$$
$$n_k \leq n'_k \qquad \text{if} \quad \sum n'_k \leq b + 3c.$$

Proof. If $b + 3c > \sum n'_k$, then we can take $n_j = 2$, which implies the claim. On the other hand, if $b + 3c \leq \sum n'_k$, then the existence of such a sequence is equivalent to the $(\sum n'_k - 3c, c)$-reachability of $(n'_1, n'_2, \ldots, n'_j)$, so the result follows from Lemma 3.8. $\square$

Lemma 3.10. *Let $S \subset \mathbb{P}^2$ be a general set of points subject to the restriction that some subset with cardinality $3c + 3$ of S is a conic collection of points. Then*

$$H^0(\mathcal{I}_{S/\mathbb{P}^2}(m)) = 0 \quad \textit{when } \#S \geq \binom{m+2}{2},$$
$$H^1(\mathcal{I}_{S/\mathbb{P}^2}(m)) = 0 \quad \textit{when } \#S \leq \binom{m+2}{2},$$

unless we have

$$\#S = 6, \quad c = 1, \quad \textit{and} \quad m = 2.$$

Proof. Write $j = \lfloor m/2 \rfloor + 1$. Because $2m - 2 + \sum_{k=2}^{j}(2m + 5 - 4k) = \binom{m+2}{2} - 3$ and $(\#S, c, m) \neq (6, 1, 2)$, Lemma 3.9 gives the existence of a $(\#S - 3 - 3c, c)$-reachable sequence $(n_1, n_2, \ldots, n_j)$ such that

$$n_1 \geq 2m - 2 \quad \text{and} \quad n_k \geq 2m + 5 - 4k \quad \text{for } k > 1 \qquad \text{if} \quad \#S \geq \binom{m+2}{2},$$
$$n_1 \leq 2m - 2 \quad \text{and} \quad n_k \leq 2m + 5 - 4k \quad \text{for } k > 1 \qquad \text{if} \quad \#S \leq \binom{m+2}{2}.$$

Now let

$$i = \begin{cases} 0 & \text{if } \#S \geq \binom{m+2}{2}, \\ 1 & \text{if } \#S \leq \binom{m+2}{2}; \end{cases}$$

so we want to show $H^i(\mathcal{I}_{S/\mathbb{P}^2}(m)) = 0$. Let S^0 be as in Lemma 3.7, and define

$$S_k = (\{q_1, q_2, q_3\} \cup S^0) \cap Q_k \smallsetminus (Q_1 \cup Q_2 \cup \cdots \cup Q_{k-1}) \quad \text{and} \quad T_k = S_k \cup S_{k+1} \cup \cdots \cup S_j.$$

We claim that $H^i(\mathcal{I}_{T_k/\mathbb{P}^2}(m + 2 - 2k)) = 0$ for all k; taking $k = 1$ will complete the proof of this lemma as $T_1 = \{q_1, q_2, q_3\} \cup S^0$.

We will prove this by backwards induction on k. For the base case $k = j$, this follows from the observation that the points of T_j are in linear general position, together with

$$\#T_j \geq \begin{cases} 1 & \text{if } m + 2 - 2j = 0 \\ 3 & \text{if } m + 2 - 2j = 1 \end{cases} \qquad \text{if } i = 0;$$
$$\#T_j \leq \begin{cases} 1 & \text{if } m + 2 - 2j = 0 \\ 3 & \text{if } m + 2 - 2j = 1 \end{cases} \qquad \text{if } i = 1.$$

For the inductive step, we first notice that $H^i(\mathcal{I}_{S_k/Q_k}(m + 2 - 2k)) = 0$ by construction. The exact sequence of sheaves

$$0 \to \mathcal{I}_{T_{k+1}/\mathbb{P}^2}(m - 2k) \to \mathcal{I}_{T_k/\mathbb{P}^2}(m + 2 - 2k) \to \mathcal{I}_{S_k/Q_k}(m + 2 - 2k) \to 0$$

gives rise to the long exact sequence in cohomology

$$\cdots \to H^i(\mathcal{I}_{T_{k+1}/\mathbb{P}^2}(m-2k)) \to H^i(\mathcal{I}_{T_k/\mathbb{P}^2}(m+2-2k)) \to H^i(\mathcal{I}_{S_k/Q_k}(m+2-2k)) \to \cdots$$

which gives the desired result by the inductive hypothesis. $\square$

Corollary 3.11. *Let $C \subset \mathbb{P}^3$ be a general BN-curve, and $H \subset \mathbb{P}^3$ be a general hyperplane. Then*

$$H^0(\mathcal{O}_H(m)) \to H^0(\mathcal{O}_{C\cap H}(m))$$

is of maximal rank, unless C is a canonically embedded curve of genus 4 and $m = 2$.

4 The Case $m = 2$

In this section, we will prove the hyperplane maximal rank theorem when $m = 2$, and the curve C is nonspecial. We will begin by constructing reducible curves with the following lemma, to which we will apply the method of Hirschowitz outlined in the introduction.

Lemma 4.1. *Let $H' \subset \mathbb{P}^r$ be a hyperplane, and (d, g) be integers with $d \geq g + r$. Assume d_1 and d_2 are positive integers with $d = d_1 + d_2$, that additionally satisfy*

$$d_1 \leq r \quad \text{and} \quad d_2 \geq r - 1.$$

Then there exists a rational normal curve $X \subset \mathbb{P}^r$ and a nonspecial BN-curve $Y \subset H'$ of degrees d_1 and d_2 respectively, with $X \cap Y$ general, such that $X \cup Y \subset \mathbb{P}^r$ is a BN-curve of genus g with $H^1(N_{X\cup Y}) = 0$.

Proof. Let $k = \min(d_1, g + 1)$. We take for Y the union of a rational normal curve R_2 of degree $r - 1$ contained in H', together with $g + 1 - k$ two-secant lines and $d_2 + k - g - r$ one-secant lines contained in H'. We take $X = L \cup R_1$ where:

1. R_1 is a rational normal curve of degree $d_1 - 1$ meeting Y in $k - 1$ points. (If $d_1 = 1$, then we take $R_1 = \emptyset$.)

2. L is a line meeting R_2 once, and meeting R_1 once (assuming $R_1 \neq \emptyset$).

By inspection, $X \cap Y$ is general and $X \cup Y$ is of genus g. To see that $X \cup Y$ is a BN-curve with $H^1(N_{X\cup Y}) = 0$, we inductively apply Lemma 2.1 to the decomposition

$$X \cup Y = (L \cup R_2) \cup R_1 \cup \{\text{secant lines to } R_2\}.$$ $\square$

Proposition 4.2. *Let $C \subset \mathbb{P}^r$ be a general BN-curve, and $H \subset \mathbb{P}^r$ be a general hyperplane. Assume that C is nonspecial. Then*

$$H^0(\mathcal{O}_H(2)) \to H^0(\mathcal{O}_{C\cap H}(2))$$

is of maximal rank.

Proof. We use induction on r; when $r = 3$, this is a consequence of Corollary 3.11. For the inductive step, we will construct a reducible curve $X \cup Y$ of degree d and genus g satisfying the conclusion of the lemma. Let (d_1, d_2) be positive integers with $d = d_1 + d_2$, such that $d_2 \geq r - 1$ and

$$\begin{aligned} d_1 = r \quad &\text{and} \quad d_2 \geq \binom{r}{2} \qquad \text{if} \quad d \geq \binom{r+1}{2}, \\ d_1 \leq r \quad &\text{and} \quad d_2 \leq \binom{r}{2} \qquad \text{if} \quad d \leq \binom{r+1}{2}. \end{aligned}$$

Pick a hyperplane H' transverse to H. By Lemma 4.1, there exists a rational normal curve $X \subset \mathbb{P}^r$ and a nonspecial BN-curve $Y \subset H'$ of degrees d_1 and d_2 respectively, with $X \cap Y$ general, such that $X \cup Y \subset \mathbb{P}^r$ is a BN-curve of genus g with $H^1(N_{X \cup Y}) = 0$. Thus, we can simultaneously generalize X and Y to a general rational normal curve and nonspecial BN-curve respectively so that $X \cup Y$ remains a BN-curve (see Lemma 2.4). Consequently, we may suppose that X and Y both satisfy the conclusion of the maximal rank theorem. Define

$$i = \begin{cases} 0 & \text{if } d \geq \binom{r+1}{2}, \\ 1 & \text{if } d \leq \binom{r+1}{2}; \end{cases}$$

so we want to show $H^i(\mathcal{I}_{(X \cup Y) \cap H/H}(2)) = 0$. By direct examination, $H^i(\mathcal{I}_{(X \cap H)/H}(1)) = 0$; and by our inductive hypothesis, $H^i(\mathcal{I}_{(Y \cap H)/(H \cap H')}(2)) = 0$. Now the long exact sequence of sheaves

$$0 \to \mathcal{I}_{(X \cap H)/H}(1) \to \mathcal{I}_{(X \cup Y) \cap H/H}(2) \to \mathcal{I}_{(Y \cap H)/(H \cap H')}(2) \to 0$$

gives rise to the long exact sequence in cohomology

$$\cdots \to H^i(\mathcal{I}_{(X \cap H)/H}(1)) \to H^i(\mathcal{I}_{(X \cup Y) \cap H/H}(2)) \to H^i(\mathcal{I}_{(Y \cap H)/(H \cap H')}(2)) \to \cdots .$$

Consequently, $H^i(\mathcal{I}_{(X \cup Y) \cap H/H}(2)) = 0$, as desired. $\square$

4.1 The Condition $d \geq g + r$

The condition $d \geq r$ is necessary; indeed when $d < g + r$, the map will sometimes fail to be of maximal rank, as shown by the following proposition:

Proposition 4.3. *Let $C \subset \mathbb{P}^r$ be any curve of degree d and genus g, with $d < g + r$ and $4d - 2g < r(r + 3)$. Then the restriction map*

$$H^0(\mathcal{O}_H(2)) \to H^0(\mathcal{O}_{C \cap H}(2))$$

fails to be of maximal rank.

Proof. We compute

$$\dim H^0(\mathcal{O}_{\mathbb{P}^r}(2)) - \dim H^0(\mathcal{O}_C(2)) = \binom{r+2}{2} - (2d + 1 - g) = \frac{r(r+3) - (4d - 2g)}{2} > 0,$$

and so C lies on a quadric. Moreover, we have

$$\begin{aligned}\dim H^0(\mathcal{O}_{\mathbb{P}^r}(2)) - \dim H^0(\mathcal{O}_C(2)) &= \frac{r(r+3) - (4d-2g)}{2} \\ &= \binom{r+1}{2} - d + (g+r-d) \\ &= \dim H^0(\mathcal{O}_H(2)) - \dim H^0(\mathcal{O}_{C\cap H}(2)) + (g+r-d) \\ &> \dim H^0(\mathcal{O}_H(2)) - \dim H^0(\mathcal{O}_{C\cap H}(2)).\end{aligned}$$

Now every quadric containing C restricts to a quadric in H containing $H \cap C$; as C is nondegenerate, this restriction has no kernel. Consequently, there is a subspace of $H^0(\mathcal{O}_H(2))$ in the kernel of $H^0(\mathcal{O}_H(2)) \to H^0(\mathcal{O}_{C\cap H}(2))$ which is of positive dimension that exceeds $\dim H^0(\mathcal{O}_H(2)) - \dim H^0(\mathcal{O}_{C\cap H}(2))$. In other words, $H^0(\mathcal{O}_H(2)) \to H^0(\mathcal{O}_{C\cap H}(2))$ is not of maximal rank. $\square$

Conjecture 4.4. The cases in Proposition 4.3 are the only cases in which the restriction map $H^0(\mathcal{O}_H(2)) \to H^0(\mathcal{O}_{C\cap H}(2))$ fails to be of maximal rank.

Conjecture 4.4 would follow from the ordinary maximal rank conjecture for $m = 2$. Indeed, if C is a general BN-curve with $d < g + r$, then C is linearly normal, i.e. $H^1(\mathcal{I}_C(1))$ vanishes. Now consider the exact sequence of sheaves

$$0 \to \mathcal{I}_C(1) \to \mathcal{O}_{\mathbb{P}^r}(1) \oplus \mathcal{I}_C(2) \to \mathcal{I}_{C\cap H}(2) \to 0;$$

this induces a long exact sequence of cohomology groups:

$$\cdots \to H^0(\mathcal{O}_{\mathbb{P}^r}(1)) \oplus H^0(\mathcal{I}_C(2)) \to H^0(\mathcal{I}_{C\cap H}(2)) \to H^1(\mathcal{I}_C(1)) \to \cdots$$

It follows that $H^0(\mathcal{O}_{\mathbb{P}^r}(1)) \oplus H^0(\mathcal{I}_C(2)) \to H^0(\mathcal{I}_{C\cap H}(2))$ is surjective, i.e. every quadric $Q \subset H$ containing $C \cap H$ is the intersection with H of a quadric $\widetilde{Q} \subset \mathbb{P}^r$ containing C. For $4d - 2g \geq r(r+3)$, the maximal rank conjecture would imply that C is not contained in any quadric, and consequently that $C \cap H$ is not contained in any quadric.

5 Construction of Reducible Curves

In this section, which is the heart of the proof, we will construct examples of reducible BN-curves $X \cup Y$ where $Y \subset H'$. These reducible curves will be the essential ingredient in applying the inductive method of Hirschowitz in the following section to deduce the hyperplane maximal rank theorem.

Lemma 5.1. *Let $H' \subset \mathbb{P}^r$ be a hyperplane, and (d, g) be integers with $\rho(d, g, r) \geq 0$ and $d \geq g + r - 2$. Assume d_1 and d_2 are positive integers with $d = d_1 + d_2$, that additionally satisfy:*

$$d_1 \geq r + \max(0, g + r - d) \quad \text{and} \quad d_2 \geq r - 1.$$

Then there exist nonspecial BN-curves $X \subset \mathbb{P}^r$ and $Y \subset H'$ of degrees d_1 and d_2 respectively, with $X \cap Y$ general, such that $X \cup Y \subset \mathbb{P}^r$ is a BN-curve of genus g with $H^1(N_{X\cup Y}) = 0$.

Proof. We will argue by induction on d and $\rho(d, g, r)$. Notice that our inequalities for d_1 and d_2 imply $d \geq 2r - 1$; for the base case, we consider when $d = 2r - 1$ or $\rho(d, g, r) = 0$.

If $d = 2r - 1$, we take X to be a rational normal curve of degree r, and $Y \subset H$ to be a rational normal of degree $r - 1$ that meets $X \cap H$ in $g + 1$ points. (Note that as $\rho(2r - 1, g, r) \geq 0$, we have $g + 1 \leq r$.) By inspection, $X \cup Y$ is of genus g; as $\operatorname{Aut} H$ acts $(r + 1)$-transitively on points in linear general position, $X \cap Y$ is general. Moreover, $X \cup Y$ is a BN-curve with $H^1(N_{X \cup Y}) = 0$ by Lemma 2.1.

If $\rho(d, g, r) = 0$ and $d \geq g + r - 2$, then either $(d, g) = (2r, r+1)$ or $(d, g) = (3r, 2r+2)$. In the case $(d, g) = (2r, r+1)$, we take X to be the union of a rational normal curve R of degree r with a 2-secant line L, and Y to be a rational normal curve of degree $r - 1$ passing through $X \cap H$. Again, by inspection $X \cup Y$ is of genus $r + 1$; as $\operatorname{Aut} H$ acts $(r + 1)$-transitively on points in linear general position, $X \cap Y$ is general. To see that $X \cup Y$ is a BN-curve with $H^1(N_{X \cup Y}) = 0$, we apply Lemma 2.1 to the decomposition $X \cup Y = (Y \cup L) \cup R$.

Now suppose that $(d, g) = (3r, 2r + 2)$. If $d_2 = r - 1$, then we take $X = C \cup L$ to be the union of a canonical curve C with a general 1-secant line L. We take Y to be the rational normal curve of degree $r - 1$ passing through $L \cap H'$ and through $r + 1$ points of $C \cap H'$. By inspection $X \cup Y$ is of genus $2r + 2$. To see that $X \cap Y$ is general, first note that since $\operatorname{Aut} H$ acts $(r + 1)$-transitively on points in linear general position, $C \cap Y$ is general; moreover, $L \cap H$ is general with respect to C. To see that $X \cup Y$ is a BN-curve, we apply Lemma 2.1 to the decomposition $X \cup Y = C \cup (L \cup Y)$, while noting that $L \cup Y$ is the specialization of a rational normal curve of degree r. Moreover, by Lemma 2.2, we have $H^1(N_{X \cup Y}) = 0$.

Otherwise, we have $d_2 \geq r$ and $d_1 \geq r + 2$; in this case we take $X = R_1 \cup L_0 \cup L_1 \cup N_1$ and $Y = R_2 \cup L_2 \cup N_2$, where:

1. R_1 is a general rational normal curve of degree r.
2. L_0 is a general 2-secant line to R_1.
3. R_2 is a general rational normal curve of degree $r - 1$ passing through all $r + 1$ points of $(R_1 \cup L_0) \cap H$.
4. L_1 is a general line meeting R_1 once and L_0 once.
5. L_2 is a general 2-secant line to R_2, passing through $L_1 \cap H$.
6. N_1 is a general rational normal curve of degree $d_1 - r - 2$ meeting L_1 once and R_1 in $d_1 - r - 2$ points (we take $N_1 = \emptyset$ if $d_1 = r + 2$).
7. N_2 is a general rational normal curve of degree $d_2 - r$ meeting L_2 once and R_2 in $d_2 - r$ points (we take $N_2 = \emptyset$ if $d_2 = r$).

In order for this to make sense, we need conditions 4 and 5 to be consistent. The consistency of 4 and 5, as well as the assertion that $X \cap Y$ is general, both follow from the following two claims:

- $L_1 \cap H$ is general relative to $(R_1 \cup L_0) \cap H$. This follows from $L_1 \cap R_1$ being general relative to L_0 and $R_1 \cap H$, which in turn follows from the existence of a rational normal curve of degree r through a general collection of $r + 3$ points.

- The 2-secant lines to R_2 sweep out H as we vary R_2 over all rational normal curves of degree $r-1$ passing through all $r+1$ points of $(R_1 \cup L_0) \cap H$. This follows from the observation that R_2 sweeps out H, which again follows from the existence of a rational normal curve of degree $r-1$ through a general collection of $r+2$ points in H'.

By inspection, $X \cup Y$ is a curve of genus g and X and Y are nonspecial. To show that $X \cup Y$ is a BN-curve, we apply Lemma 2.1 to the decomposition

$$X \cup Y = (L_0 \cup R_2) \cup R_1 \cup (L_1 \cup L_2 \cup N_1 \cup N_2).$$

Similarly, to show $H^1(N_{X \cup Y}) = 0$, we apply Lemma 2.1 and then Lemma 2.3 to the decomposition

$$X \cup Y = (L_0 \cup R_2) \cup R_1 \cup L_2 \cup N_1 \cup N_2 \cup L_1.$$

To apply Lemma 2.3, we need to check that the tangent lines to $(L_0 \cup R_2) \cup R_1 \cup L_2 \cup N_1 \cup N_2$ at the points of intersection with L_1 do not all lie in a plane. Since L_1 intersects L_0, the only possible plane that could contain all 3 tangents is $\overline{L_0 L_1}$. But as this plane contains the two points of intersection of L_0 with R_1 and a plane can only intersect a rational normal curve at 3 points with multiplicity, the tangent line to R_1 at $L_1 \cap R_1$ cannot be contained in this plane. Consequently, we may apply Lemma 2.3 as claimed.

For the inductive step, we have $d \geq 2r$ and $\rho(d, g, r) > 0$. We claim that these inequalities imply that

$$r + \max(0, g + r - d) + r - 1 < d = d_1 + d_2. \tag{1}$$

Of course,

$$r + \max(0, g + r - d) + r - 1 = \max(2r - 1, 3r - 1 + g - d);$$

consequently, as $2r - 1 < 2r \leq d$, it suffices to show $3r - 1 + g - d < d$, or equivalently $g < 2d + 1 - 3r$. To see this, note that if $g \geq 2d + 1 - 3r$, then we would have

$$-(r-1)(d-2r) = (r+1)d - r(2d + 1 - 3r) - r(r+1) \geq (r+1)d - rg - r(r+1) > 0,$$

which is a contradiction; thus, $g < 2d + 1 - 3r$, and so (1) holds. Consequently, there exists (d_1', d_2') either equal to $(d_1 - 1, d_2)$ or to $(d_1, d_2 - 1)$, such that $d_1' \geq r + \max(0, g + r - d)$ and $d_2' \geq r - 1$. (Otherwise $d_1 - 1 < r + \max(0, g + r - d)$ and $d_2 - 1 < r - 1$, i.e. $d_1 \leq r + \max(0, g + r - d)$ and $d_2 \leq r - 1$; adding these contradicts (1).)

If we define $g' = \max(0, g - 1)$, then $\max(0, g + r - d) = \max(0, g' + r - (d - 1))$. Thus by the inductive hypothesis, there are BN-curves $X' \subset \mathbb{P}^r$ and $Y' \subset H'$ of degrees d_1' and d_2' respectively, with $X' \cap Y'$ general, such that $X' \cup Y' \subset \mathbb{P}^r$ is a BN-curve of genus g' with $H^1(N_{X' \cup Y'}) = 0$. To complete the inductive step, we take

$$(X, Y) = \begin{cases} (X', Y' \cup L) & \text{if } d_1' = d_1; \\ (X' \cup L, Y') & \text{if } d_2' = d_2; \end{cases} \quad \text{where} \quad L = \begin{cases} \text{a 1-secant line} & \text{if } g' = g; \\ \text{a 2-secant line} & \text{if } g' \neq g. \end{cases}$$

This satisfies the desired conclusion by Lemma 2.1. $\square$

Lemma 5.2. *Let $H' \subset \mathbb{P}^r$ be a hyperplane, and (d, g) be integers with $\rho(d, g, r) \geq 0$. Assume d_1 and d_2 are positive integers with $d = d_1 + d_2$, that additionally satisfy:*

$$d_1 \geq r + \max(0, g + r - d) \quad \text{and} \quad d_2 \geq r - 1.$$

Then there exists BN-curves $X \subset \mathbb{P}^r$ and $Y \subset H'$ of degrees d_1 and d_2 respectively, with $X \cap Y$ general, such that $X \cup Y \subset \mathbb{P}^r$ is a BN-curve of genus g with $H^1(N_{X\cup Y}) = 0$. Moreover, we can take X to be nonspecial if

$$d_2 \geq (r-1) \cdot \left\lceil \frac{\max(0, g + r - d)}{2} \right\rceil. \tag{2}$$

Proof. We will argue by induction on d. When $d \geq g + r - 2$, we are done by Lemma 5.1. Thus we may assume that $d < g + r - 2$. In particular, this implies that $d \geq 4r$, and that $\max(0, g + r - d) = g + r - d$. We claim that

$$r + \max(0, g + r - d) + r - 1 = 3r - 1 + g - d < d - 2(r - 2) = d_1 + d_2 - 2(r - 2). \tag{3}$$

This is equivalent to $g < 2d + 5 - 5r$; to see this, note that if $g \geq 2d + 5 - 5r$, then

$$-(r-1)(d-4r) - 2r = (r+1)d - r(2d + 5 - 5r) - r(r+1) \geq (r+1)d - rg - r(r+1) = 0,$$

which is a contradiction; thus, $g < 2d + 5 - 5r$, and so (3) holds. Consequently, there exists (d_1', d_2') either equal to $(d_1 - 1, d_2 - r + 1)$ or to $(d_1 - r, d_2)$, such that

$$d_1' \geq r + \max(0, g + r - d) - 1 = r + \max(0, (g - r - 1) + r - (d - r))$$

and $d_2' \geq r - 1$. (Otherwise $d_1 - r < r + \max(0, g + r - d) - 1$ and $d_2 - r + 1 < r - 1$, i.e. $d_1 - (r - 2) \leq r + \max(0, g + r - d)$ and $d_2 - (r - 2) \leq r - 1$; adding these contradicts (3).)

Thus by the inductive hypothesis, there are BN-curves $X' \subset \mathbb{P}^r$ and $Y' \subset H'$ of degrees d_1' and d_2' respectively, with $X' \cap Y'$ general, such that $X' \cup Y' \subset \mathbb{P}^r$ is a BN-curve of genus $g - r - 1$ with $H^1(N_{X'\cup Y'}) = 0$. To complete the inductive step, we take

$$(X, Y) = \begin{cases} (X' \cup L, Y' \cup R_2) & \text{if } d_1' = d_1 - 1; \\ (X' \cup R_1, Y') & \text{if } d_2' = d_2. \end{cases}$$

Here, R_1 is a rational normal curve of degree r that is $(r+2)$-secant to X', and L is a 1-secant line to X', and R_2 is a rational normal curve of degree $r - 1$ intersecting Y' in $r + 1$ points and passing through $L \cap H$.

Tracing through the proof, we notice then when (2) is satisfied, we add a 1-secant line to X at least as many times as we add an $(r + 2)$-secant rational normal curve of degree r. In particular, when (2) holds, the curve X we constructed is nonspecial. □

6 The Inductive Argument

In this section, we combine the results of the previous three sections to inductively prove the hyperplane maximal rank theorem. This essentially boils down to manipulating inequalities to show that we can choose the integers (d_1, d_2) appearing in the previous section in the appropriate fashion.

We begin by giving some bounds on the expressions appearing in Lemma 5.2 that are easier to manipulate.

Lemma 6.1. *Let d, g, and r be integers with $\rho(d,g,r) \geq 0$. Then*

$$r + \max(0, g+r-d) \leq r - 1 + \frac{d}{r} \quad \text{and} \quad (r-1) \cdot \left\lceil \frac{\max(0, g+r-d)}{2} \right\rceil \leq \frac{r-1}{2r} \cdot d.$$

Proof. By assumption,

$$r \cdot (g+r-d) \leq r \cdot (g+r-d) + (r+1)d - rg - r(r+1) = d - r$$

$$\Rightarrow \max(0, g+r-d) \leq \frac{d-r}{r}.$$

Substituting this in, we find

$$r + \max(0, g+r-d) \leq r + \frac{d-r}{r} = r - 1 + \frac{d}{r}$$
$$\left\lceil \frac{\max(0, g+r-d)}{2} \right\rceil \leq \frac{\frac{d-r}{r} + 1}{2} = \frac{r-1}{2r} \cdot d.$$

□

Lemma 6.2. *Let d, r, and m be integers with*

$$r \geq 4, \quad m \geq 3, \quad \text{and} \quad d \geq 2r + 2.$$

Assume that

$$d \geq \binom{m+r-1}{m}, \quad \text{respectively} \quad d \leq \binom{m+r-1}{m}.$$

Then there are integers d_1 and d_2 such that $d = d_1 + d_2$ and

$$d_1 \geq \binom{m+r-2}{m-1} \quad \text{and} \quad d_2 \geq \binom{m+r-2}{m},$$

$$\text{respectively} \quad d_1 \leq \binom{m+r-2}{m-1} \quad \text{and} \quad d_2 \leq \binom{m+r-2}{m},$$

which moreover satisfy

$$d_1 \geq r - 1 + \frac{d}{r} \quad \text{and} \quad d_2 \geq r - 1.$$

Additionally, if $m = 3$, we can replace $d_2 \geq r-1$ by the stronger assumption that

$$d_2 \geq \frac{r-1}{2r} \cdot d.$$

Proof. First we consider the case where

$$d = \binom{m+r-1}{m} \geq \binom{r+2}{3} \geq 2r+2.$$

In this case, we take

$$d_1 = \binom{m+r-2}{m-1} \quad \text{and} \quad d_2 = \binom{m+r-2}{m}.$$

To see that these satisfy the given conditions, first note that

$$d_2 \geq \binom{r+1}{3} \geq r-1.$$

Next note that

$$\binom{m+r-1}{m} \geq \frac{r-1}{\frac{m}{m+r-1} - \frac{1}{r}};$$

indeed, the LHS is an increasing function of m, the RHS is a decreasing function of m, and the inequality is obvious for $m = 3$. Rearranging, we get

$$d_1 = \binom{m+r-2}{m-1} = \frac{m}{m+r-1} \cdot \binom{m+r-1}{m} \geq r-1+\frac{1}{r} \cdot \binom{m+r-1}{m} = r-1+\frac{d}{r}.$$

If $m = 3$, then

$$d_2 = \binom{r+1}{3} \geq \frac{r-1}{2r} \cdot \binom{r+2}{3} = \frac{r-1}{2r} \cdot d.$$

In general, we induct upwards on d in the $\geq$ case and downwards on d in the $\leq$ case. To do this, we want to show that if d_1 and d_2 satisfy

$$d_1 \geq r-1+\frac{d}{r} \quad \text{and} \quad d_2 \geq \frac{r-1}{2r} \cdot d \quad \text{where} \quad d = d_1 + d_2 \geq 2r+2,$$

then either $(d_1 - 1, d_2)$ or $(d_1, d_2 - 1)$, as well as either $(d_1 + 1, d_2)$ or $(d_1, d_2 + 1)$, satisfy the above two conditions. We note that

$$\begin{aligned} d_1 \geq r-1+\frac{d}{r} = r-1+\frac{d_1+d_2}{r} \quad &\Leftrightarrow \quad (r-1)d_1 \geq r(r-1)+d_2. \\ d_2 \geq \frac{r-1}{2r} \cdot d = \frac{r-1}{2r} \cdot (d_1+d_2) \quad &\Leftrightarrow \quad (r+1)d_2 \geq (r-1)d_1. \end{aligned}$$

Assume (to the contrary) that neither $(d_1 - 1, d_2)$ nor $(d_1, d_2 - 1)$ satisfy the conditions, respectively that neither $(d_1 + 1, d_2)$ nor $(d_1, d_2 + 1)$ satisfy the conditions. Then we must have

$$\begin{aligned} &(r-1)(d_1-1) < r(r-1)+d_2 \quad \text{and} \quad (r+1)(d_2-1) < (r-1)d_1, \\ \text{respectively} \quad &(r-1)d_1 < r(r-1)+d_2+1 \quad \text{and} \quad (r+1)d_2 < (r-1)(d_1+1). \end{aligned}$$

Equivalently, we must have

$$(r-1)(d_1-1)+1 \le r(r-1)+d_2 \quad \text{and} \quad (r+1)(d_2-1)+1 \le (r-1)d_1,$$
$$\text{respectively} \quad (r-1)d_1 \le r(r-1)+d_2 \quad \text{and} \quad (r+1)d_2+1 \le (r-1)(d_1+1).$$

Adding twice the first equation to the second, we must have

$$2(r-1)(d_1-1)+2+(r+1)(d_2-1)+1 \le 2r(r-1)+2d_2+(r-1)d_1,$$
$$\text{respectively} \quad 2(r-1)d_1+(r+1)d_2+1 \le 2r(r-1)+2d_2+(r-1)(d_1+1).$$

Simplifying yields

$$(r-1)(d_1+d_2) \le 2r^2+r-4, \quad \text{respectively} \quad (r-1)(d_1+d_2) \le 2r^2-r-2.$$

In particular,

$$d = d_1+d_2 \le \frac{2r^2+r-4}{r-1} = 2r+3-\frac{1}{r-1} \quad \Rightarrow \quad d \le 2r+2.$$

Consequently, we can reach via upward and downward induction every value of d that is at least $2r+2$. $\square$

Proof of the Hyperplane Maximal Rank Theorem. We use induction on m and r. For $m=2$, this is a consequence of Proposition 4.2; for $r=3$, this is a consequence of Corollary 3.11. Note that if $d \le 2r-1$, then C is nonspecial and so $H^0(\mathcal{O}_H(2)) \to H^0(\mathcal{O}_{C\cap H}(2))$ is surjective; consequently, $H^0(\mathcal{O}_H(m)) \to H^0(\mathcal{O}_{C\cap H}(m))$ is surjective for all $m \ge 2$. Thus, we may suppose $d \ge 2r$.

For the inductive step, we define integers (d_1, d_2) as follows. If $d \in \{2r, 2r+1\}$, we take $(d_1, d_2) = (r+1, d-r-1)$. Otherwise, for $d \ge 2r+2$, we let (d_1, d_2) be as in Lemma 6.2. Fix another hyperplane H' transverse to H. By Lemma 5.2, plus Lemma 6.1 when $d \ge 2r+2$, there exist BN-curves $X \subset \mathbb{P}^r$ and $Y \subset H'$ of degrees d_1 and d_2 respectively, with $X \cap Y$ general, such that $X \cup Y \subset \mathbb{P}^r$ is a BN-curve of genus g with $H^1(N_{X\cup Y}) = 0$. Moreover, if $m=3$, then we can arrange for X to be nonspecial.

We simultaneously generalize both X and Y to BN-curves that satisfy the hyperplane maximal rank conjecture while keeping $X \cup Y$ a BN-curve (see Lemma 2.4). Define

$$i = \begin{cases} 0 & \text{if } d \ge \binom{r+m-1}{m}, \\ 1 & \text{if } d \le \binom{r+m-1}{m}; \end{cases}$$

so we want to show $H^i(\mathcal{I}_{(X\cup Y)\cap H}(m)) = 0$. The exact sequence of sheaves

$$0 \to \mathcal{I}_{(X\cap H)/H}(m-1) \to \mathcal{I}_{(X\cup Y)\cap H/H}(m) \to \mathcal{I}_{(Y\cap H)/(H\cap H')}(m) \to 0,$$

gives rise to a long exact sequence in cohomology

$$\cdots \to H^i(\mathcal{I}_{(X\cap H)/H}(m-1)) \to H^i(\mathcal{I}_{(X\cup Y)\cap H/H}(m)) \to H^i(\mathcal{I}_{(Y\cap H)/(H\cap H')}(m)) \to \cdots.$$

By the inductive hypothesis, we have $H^i(\mathcal{I}_{(X\cap H)/H}(m-1)) = H^i(\mathcal{I}_{(Y\cap H)/(H\cap H')}(m)) = 0$. Consequently, $H^i(\mathcal{I}_{(X\cup Y)\cap H/H}(m)) = 0$, as desired. $\square$

7 Quadric Sections of Space Curves

In this section, we will apply and expand the techniques developed in Section 3 to study quadric sections of space curves. Let $C \subset \mathbb{P}^3$ be a general BN-curve, and $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ be a general (smooth) quadric. Here, we study the restriction maps

$$H^0(\mathcal{O}_Q(m, n)) \to H^0(\mathcal{O}_{C \cap Q}(m, n)).$$

In this section, there are proofs analogous to proofs given in Section 3; we will indicate when this is the case so that the impatient reader may skip or skim them as desired.

As in Section 3, we will specialize to the case where C is a defining curve; we will then compute the quadric sections of defining curves. In order to do this, we first need the following lemma.

Lemma 7.1. *Let $E \subset Q$ be an elliptic normal curve, and $\{q_1, q_2, \ldots, q_6\}$ be six general points on E. Write C for the unique rational normal curve of degree 3 through $\{q_1, q_2, \ldots, q_6\}$.*

1. *If $\{p_1, p_2\} \subset E$ satisfy $q_1 + q_2 + \cdots + q_6 \sim H + p_1 + p_2$, then $\#(C \cap D) = 1$ where D is the unique line through $\{p_1, p_2\}$.*
2. *If $\{p_1, p_2\} \subset E$ satisfy $q_1 + q_2 + \cdots + q_6 + p_1 + p_2 \sim 3H$, then $\#(C \cap D) = 2$ where D is the unique line through $\{p_1, p_2\}$.*
3. *If $\{p_1, p_2, \ldots, p_6\} \subset E$ satisfy $q_1 + q_2 + \cdots + q_6 + p_1 + p_2 + \cdots + p_6 \sim 3H$, then $\#(C \cap D) = 5$ where D is the unique rational normal curve of degree 3 through $\{p_1, p_2, \ldots, p_6\}$.*

Proof. The families of 1-secant lines, 2-secant lines, and 5-secant rational normal curves of degree 3 to C, are irreducible families of dimensions 3, 2, and 7 respectively.

First, we will prove that families of sets $\{p_1, \ldots\}$ satisfying the three above conditions for some E are also irreducible of dimensions 3, 2, and 7 respectively. For this, consider the incidence correspondences $\{(\text{such a collection of points, such a curve } E)\}$. These incidence correspondences dominate these families, with generic fibers of dimensions 0, 1, and 0 respectively; thus it suffices to show that these incidence correspondences are irreducible of dimensions 3, 3, and 7 respectively. But the projection maps from each of these incidence correspondences onto the family of elliptic normal curves through $\{q_1, q_2, \ldots, q_6\}$ are flat with irreducible and equidimensional fibers of dimensions 1, 1, and 5 respectively. To finish the proof of this claim, note that the family of elliptic normal curves through $\{q_1, q_2, \ldots, q_6\}$ is irreducible of dimension 2.

Consequently, it suffices to show the converse of each of the above statements. But the converses to the above statements follow from the following 3 facts respectively.

1. If D is a general 1-secant line to C, then $C \cup D$ is contained in a quadric Q', on which C and D are curves of types $(1, 2)$ and $(0, 1)$ respectively.
2. If D is a general 2-secant line to C, then $C \cup D$ is contained in a quadric Q', on which C and D are curves of types $(1, 2)$ and $(1, 0)$ respectively.
3. If D is a general 5-secant rational normal curve of degree 3 to C, then $C \cup D$ is contained in a quadric Q', on which C and D are curves of types $(1, 2)$ and $(2, 1)$ respectively.

Each of these facts can be proved as follows: We first compute that $\dim H^0(\mathcal{O}_{C\cup D}(2))$ is respectively 9, 8, and 9; in each case this is less than $\dim H^0(\mathcal{O}_{\mathbb{P}^3}(2)) = 10$. Thus $C \cup D$ lies on a quadric Q'. Since D is general, Q' must be smooth; indeed the smoothness of Q' is an open condition and when C and D are general curves of the above types on a smooth quadric, D is a 1-secant line, 2-secant line, or 5-secant rational normal curve of degree 3, respectively. Finally, intersection theory on $Q' \simeq \mathbb{P}^1 \times \mathbb{P}^1$ forces the types of C and D to be as claimed. $\square$

This prompts the following definition:

Definition 7.2. An *elliptic collection* with *signature* (a, b, c) is a set of $2a + 2b + 6(c + 1)$ points $\{q_1, q_2, \ldots, q_6, p_1, p_2, \ldots, p_{2a+2b+6c}\} \subset Q$ such that

1. $q_1 + q_2 + \cdots + q_6 \sim p_{2k+1} + p_{2k+2}$ for $0 \leq k \leq a - 1$.
2. $q_1 + q_2 + \cdots + q_6 + p_{2a+2k+1} + p_{2a+2k+2} \sim 2H$ for $0 \leq k \leq b - 1$
3. $q_1 + q_2 + \cdots + q_6 + p_{2a+2b+6k+1} + p_{2a+2b+6k+2} + \cdots + p_{2a+2b+6k+6} \sim 3H$ for $0 \leq k \leq c - 1$.

Corollary 7.3. *Let $S \subset Q$ be a general elliptic collection of $2a + 2b + 6(c + 1)$ points with signature (a, b, c). Then there is a defining curve of signature (a, b, c) whose intersection with Q is S.*

Proof. This follows from Lemma 7.1. $\square$

Lemma 7.4. *Let a, b, and c be integers, and $E_1, E_2, \ldots, E_j$ be general elliptic normal curves on Q passing through $\{q_1, q_2, \ldots, q_6\}$. If $S = \{p_1, p_2, \ldots, p_{2a+2b+6c}\} \subset Q$ is a general set of $2a + 2b + 6c$ points subject to the constraint that $\{q_1, q_2, \ldots, q_6, p_1, p_2, \ldots, p_{2a+2b+6c}\}$ is an elliptic collection of signature (a, b, c), and $(n_1, n_2, \ldots, n_j)$ is $(a+b, c)$-reachable, then we can specialize S to a subscheme $S^0 \subset Q$ with*

$$\deg S_k^0 = 2n_k \quad \textit{where} \quad S_k^0 = S^0 \cap E_k \smallsetminus (E_1 \cup E_2 \cup \cdots \cup E_{k-1}),$$

and moreover the divisor class of S_k^0 on E_k is an integral linear combination of $\{q_1, q_2, \ldots, q_6\}$ and the hyperplane class H.

Proof. (This follows, mutatis mutandis, from the same argument as in Lemma 3.7.)

We use induction on c; the base case $c = 0$ is obvious. When we increase c by one, we add 6 points $\{p_2, p_2', p_1, p_1', p_0, p_0'\}$ lying on a general elliptic normal curve $E \subset Q$ passing through $\{q_1, q_2, \ldots, q_6\}$, and we add three dots to the columns (i_2, i_1, i_0), respectively (i_2, i_0), respectively (i_0). In the first case, we begin by specializing $\{p_1, p_1'\}$ and $\{p_2, p_2'\}$ to the remaining points of intersection of E with E_{i_1} and E_{i_2} respectively; similarly, in the second case, we begin by specializing $\{p_2, p_2'\}$ to the remaining point of intersection of E with E_{i_2}. After this, we specialize E to E_{i_0} while preserving these incidence relations. $\square$

Lemma 7.5. *Let $E \subset Q$ be a general elliptic normal curve on Q, and ℓ be a nonzero integer. Then the line bundle $\mathcal{O}_E(\ell, -\ell)$ is nontrivial.*

Proof. It suffices to show that for every elliptic curve E and $p, q \in E$, there is an embedding $E \hookrightarrow Q$ as an elliptic normal curve with $\mathcal{O}_E(1, -1) = \mathcal{O}_E(2p - 2q)$. But this follows from the fact that for every $z \in E$, there is a quadratic covering $E \to \mathbb{P}^1$ ramified at z. $\square$

Corollary 7.6. *Let S_k^0 be as in Lemma 7.4, and define*

$$S_k = \begin{cases} \{q_1, q_2, \ldots, q_6\} \cup S_1^0 & \text{if } k = 1; \\ S_k^0 & \text{otherwise.} \end{cases}$$

Then for any integers $m' \neq n'$, we have $\mathcal{O}_E(S_k) \not\simeq \mathcal{O}_E(m', n')$.

Lemma 7.7. *Let $S \subset Q$ be a general elliptic collection of points of signature (a, b, c) and $m < n$ be nonnegative integers. Then*

$$H^0(\mathcal{I}_{S/Q}(m, n)) = 0 \quad \text{when } \#S \geq (m+1)(n+1),$$
$$H^1(\mathcal{I}_{S/Q}(m, n)) = 0 \quad \text{when } \#S \leq (m+1)(n+1),$$

unless we have

$$\#S = 12, \quad c = 1, \quad \text{and} \quad (m, n) = (2, 3).$$

Proof. (The proof proceeds, mutatis mutandis, in a similar fashion to Lemma 3.10.) Write $j = \lfloor m/2 \rfloor + 1$, and define

$$f(m, n, k) = \begin{cases} m + n - 3 & \text{if } k = 1, \\ \left\lfloor \frac{n-m+1}{2} \right\rfloor & \text{if } m = 2(k-1) \text{ and } \#S \leq (m+1)(n+1), \\ \left\lceil \frac{n-m+1}{2} \right\rceil & \text{if } m = 2(k-1) \text{ and } \#S \geq (m+1)(n+1), \\ m + n + 4 - 4k & \text{otherwise.} \end{cases}$$

Because $\sum_{k=1}^{j} f(m, n, k) = (m+1)(n+1) - 3$ and $(\#S, c, m, n) \neq (12, 1, 2, 3)$, Lemma 3.9 gives the existence of a $(\#S - 3 - 3c, c)$-reachable $(n_1, n_2, \ldots, n_j)$ such that

$$n_k \geq f(m, n, k) \qquad \text{if} \quad \#S \geq (m+1)(n+1);$$
$$n_k \leq f(m, n, k) \qquad \text{if} \quad \#S \leq (m+1)(n+1).$$

Now let

$$i = \begin{cases} 0 & \text{if } \#S \geq (m+1)(n+1), \\ 1 & \text{if } \#S \leq (m+1)(n+1); \end{cases}$$

so we want to show $H^i(\mathcal{I}_{S/Q}(m, n)) = 0$. We let S_k be as in Lemma 7.6, and we define $T_k = S_k \cup S_{k+1} \cup \cdots \cup S_j$. We claim that $H^i(\mathcal{I}_{T_k/Q}(m + 2 - 2k, n + 2 - 2k)) = 0$ for all k; taking $k = 1$ will complete the proof of this lemma as $T_1 = \{q_1, q_2, \ldots, q_6\} \cup S^0$.

We will prove this by backwards induction on k. For the base case $k = j$, note that $T_j \subset E_j$ and $\mathcal{O}_{E_j}(T_j) \neq \mathcal{O}_{E_j}(1, n - m + 1)$ by Lemma 7.6. It thus suffices to observe that

$$\#T_j \geq \begin{cases} n - m + 1 & \text{if } m + 2 - 2j = 0 \text{ and } n + 2 - 2j = n - m \\ 2(n - m + 2) & \text{if } m + 2 - 2j = 1 \text{ and } n + 2 - 2j = n - m + 1 \end{cases} \qquad \text{if } i = 0;$$

$$\#T_j \leq \begin{cases} n - m + 1 & \text{if } m + 2 - 2j = 0 \text{ and } n + 2 - 2j = n - m \\ 2(n - m + 2) & \text{if } m + 2 - 2j = 1 \text{ and } n + 2 - 2j = n - m + 1 \end{cases} \qquad \text{if } i = 1.$$

For the inductive step, we first notice that $H^i(\mathcal{I}_{S_k/E_k}(m+2-2k, n+2-2k)) = 0$ by Lemma 7.6. The exact sequence of sheaves

$$0 \to \mathcal{I}_{T_{k+1}/Q}(m-2k, n-2k) \to \mathcal{I}_{T_k/Q}(m+2-2k, n+2-2k) \to \mathcal{I}_{S_k/E_k}(m+2-2k, n+2-2k) \to 0$$

gives rise to the long exact sequence in cohomology

$$\begin{aligned} \cdots \to H^i(\mathcal{I}_{T_{k+1}/Q}(m-2k, n-2k)) \\ \to H^i(\mathcal{I}_{T_k/Q}(m+2-2k, n+2-2k)) \\ \to H^i(\mathcal{I}_{S_k/E_k}(m+2-2k, n+2-2k)) \to \cdots \end{aligned}$$

which gives the desired result by the inductive hypothesis. $\square$

Corollary 7.8. *If C is a general BN-curve, $Q \subset \mathbb{P}^3$ is a general hyperplane, and (m,n) are positive integers with $m < n$, then the restriction map*

$$H^0(\mathcal{O}_Q(m,n)) \to H^0(\mathcal{O}_{C\cap Q}(m,n))$$

is of maximal rank, except when $(m,n) = (2,3)$ and C is a canonically embedded curve of genus 4.

When $m = n$, we are faced with an additional difficulty: Lemma 7.6 does not hold for $m' = n'$. To remedy this, we need the following stronger notion of reachability.

Definition 7.9. Let a, b, and c be nonnegative integers. Start with j empty columns, and consider the following game, where we perform the first step a times, the second step b times, and our choice of the remaining steps c times.

1. Pick any column and add a ✓ to it.

   ✓

2. Pick any column and add a $-$ to it.

   $-$

3. Pick any three columns and add a $-$ in the first two and a $+$ in the last.

   $-\quad -\quad +$

4. Pick any two columns and add a $-$ to the first one, and a $+$ and $-$ to the second one.

   $\begin{array}{cc} & + \\ - & - \end{array}$

5. Pick any column and add two $-$ and one $+$ to it.

   $\begin{array}{c} + \\ - \\ - \end{array}$

We say that a sequence of positive integers $(n_1, n_2, \ldots, n_j)$ is (a,b,c)-*elliptic-reachable* if we can do this so that

1. There are n_k symbols in the kth column.

2. Every column either has a ✓ in it or has a different number of $-$'s and $+$'s.

3. The first column either has a ✓ or does not have exactly one more $-$ than $+$.

Lemma 7.10. *Let a, b, and c be integers, and $E_1, E_2, \ldots, E_j$ be general elliptic normal curves on Q passing through $\{q_1, q_2, \ldots, q_6\}$. Assume $S = \{p_1, p_2, \ldots, p_{2a+2b+6c}\} \subset Q$ is a general set of $2a + 2b + 6c$ points subject to the constraint that $\{q_1, q_2, \ldots, q_6, p_1, p_2, \ldots, p_{2a+2b+6c}\}$ is an elliptic collection of signature (a, b, c), and $(n_1, n_2, \ldots, n_j)$ is (a, b, c)-elliptic-reachable. Then we can specialize S to a subscheme $S^0 \subset Q$ such that $\deg S^0_k = 2n_k$ and S_k is not linearly equivalent to a multiple of the hyperplane class on E_k, where S^0_k and S_k are as in Lemmas 7.4 and 7.6.*

Proof. First, we notice that if $(n_1, n_2, \ldots, n_j)$ is (a, b, c)-elliptic-reachable, then it is also $(a-1, b+1, c)$-elliptic-reachable. Moreover, any elliptic collection of signature $(a-1, b+1, c)$ is the specialization of elliptic collections of signature (a, b, c); indeed, any 2-secant line is the specialization of 1-secant lines. Consequently, it suffices to prove this lemma in the case where $(n_1, n_2, \ldots, n_j)$ is (a, b, c)-elliptic-reachable, but is not $(a-1, b+1, c)$-elliptic-reachable. In particular, in order to prove this lemma, we can replace the ✓'s with $+$'s in the first step of the game in our definition of elliptic reachability. After making this change, we use exactly the same construction as in Lemma 7.4. □

Lemma 7.11. *Suppose that a, b, and c are nonnegative integers with either $a = 0$ or $c = 0$. Let $(n_1, n_2, \ldots, n_j)$ be a sequence with $\sum n_k = a + b + 3c$, which is of one of the following forms:*

1. $(n_1, 2m-4, 2m-8, \ldots, 4, 1)$, where m is even and $2m - 3 \leq n_1 \leq \frac{5m-8}{2}$.

2. $(n_1, 2m-4, 2m-8, \ldots, 4)$, where m is even and $\frac{3m-4}{2} \leq n_1 \leq 2m - 3$.

3. $(n_1, 2m-4, 2m-8, \ldots, 2)$, where m is odd and $\frac{3m-5}{2} \leq n_1 \leq \frac{5m-7}{2}$.

Then $(n_1, n_2, \ldots, n_j)$ is (a, b, c)-elliptic-reachable unless we are in one of the following cases:

$\boldsymbol{m}$	$\boldsymbol{(a, b, c)}$	$\boldsymbol{[n_1, n_2, \ldots, n_j]}$
2	$(0, 2, 1)$	$[1, 1]$
	$(0, 1, 1)$	$[1]$
3	$(0, 2, 2)$	$[3, 2]$
	$(0, 0, 3)$	$[4, 2]$
	$(0, 1, 2)$	$[2, 2]$
4	$(0, 0, 4)$	$[5, 4]$

Proof. The case $m \leq 4$ is a straight-forward finite computation, which we relegate to Appendix A.

For $m \geq 5$, we will prove a stronger statement: the (a, b, c)-elliptic-reachability of all listed sequences, in addition to the sequences

1. $(n_1, 2m-4, 2m-8, \ldots, 4, 1)$, if m is even and $n_1 \in \{2m, 2m-2, 2m-4\}$.

2. $(n_1, 2m-4, 2m-8, \ldots, 4)$, if m is even and $n_1 \in \{2m, 2m-2, 2m-4\}$.

3. $(n_1, 2m-4, 2m-8, \ldots, 2)$, if m is odd and $n_1 \in \{2m, 2m-2, 2m-4\}$.

For this, we will use induction on m. The base cases $m = 5$ and $m = 6$ are again straightforward finite computations which we relegate to Appendix A.

For the inductive step, we start by playing moves 1, 2, and 5 with the first column until we can no longer do so any more (without exceeding n_1 symbols in the first column). At this point, we must have at most two free slots left in the first column. Moreover, we must have exhausted all of the moves 1 and 2 that we are allowed, and must therefore have at least two remaining uses of moves 3/4/5 left. We then apply move 4 with the first two columns as many times as we have remaining slots in the first column. Since $n_1 \geq 8$, we must either have put a ✓ in the first column or put at least 4 more $-$'s than $+$'s. Finally, we apply the inductive hypothesis to the remaining columns (with n_2 decreased by the number of symbols we added to the second column). □

Lemma 7.12. *Let $S \subset Q$ be a general elliptic collection of points of signature (a, b, c), where either $a = 0$ or $c = 0$. Suppose that m is a positive integer. Then*

$$H^0(\mathcal{I}_{S/Q}(m,m)) = 0 \quad \textit{when } \#S \geq (m+1)^2,$$
$$H^1(\mathcal{I}_{S/Q}(m,m)) = 0 \quad \textit{when } \#S \leq (m+1)^2;$$

unless we are in one of the following cases

$\boldsymbol{m}$	$\boldsymbol{(a,b,c)}$
2	$(0,0,1)$, $(0,2,0)$, *or* $(0,1,0)$
3	$(0,0,1)$, $(0,2,1)$, *or* $(0,1,1)$

Proof. If $(a, b, c) = (0, 0, 1)$, Lemma 7.7 implies that

$$H^0(\mathcal{I}_{S/Q}(1,2)) = H^1(\mathcal{I}_{S/Q}(3,4)) = 0.$$

Consequently, $H^0(\mathcal{I}_{S/Q}(m,m)) = 0$ for $m \leq 1$ and $H^1(\mathcal{I}_{S/Q}(m,m)) = 0$ for $m \geq 4$. As our assumptions in this case imply $m \neq 2, 3$, we are done. For the rest of the proof, we assume that $(a, b, c) \neq (0, 0, 1)$.

Note that if $\#S \geq (m+1)(m+2)$, then Lemma 7.7 implies $H^0(\mathcal{I}_{S/Q}(m, m+1)) = 0$, and so $H^0(\mathcal{I}_{S/Q}(m,m)) = 0$. Similarly, if $\#S \leq m(m+1)$, then $H^1(\mathcal{I}_{S/Q}(m,m)) = 0$. Consequently, we may assume additionally that

$$m(m+1) < \#S < (m+1)(m+2).$$

Recall that $\#S$ is even; in particular if m is even then $\#S \neq (m+1)^2$. Write

$$j = \begin{cases} \frac{m}{2} + 1 & \text{if } m \text{ even and } \#S > (m+1)^2; \\ \frac{m+1}{2} & \text{if } m \text{ odd;} \\ \frac{m}{2} & \text{if } m \text{ even and } \#S < (m+1)^2. \end{cases}$$

Define the sequence $(n_1, n_2, \ldots, n_j)$ by

$$n_k = \begin{cases} 1 & \text{if } m = 2(k-1); \\ \frac{1}{2}(\#S - m^2 + 2m - 8) & \text{if } k = 1, \text{ and } m \text{ even, and } \#S > (m+1)^2; \\ \frac{1}{2}(\#S - m^2 + 2m - 7) & \text{if } k = 1 \text{ and } m \text{ odd;} \\ \frac{1}{2}(\#S - m^2 + 2m - 6) & \text{if } k = 1 \text{ and } m \text{ even, and } \#S < (m+1)^2; \\ 2m + 4 - 4k & \text{otherwise.} \end{cases}$$

Unless $m = 3$ and $(a, b, c) = (0, 0, 2)$, or $m = 4$ and $(a, b, c) = (0, 0, 3)$, we can specialize S as in Lemma 7.11; note that $H^i(\mathcal{I}_{S_k/E_k}(m + 2 - 2k, n + 2 - 2k)) = 0$, where

$$i = \begin{cases} 0 & \text{if } \#S \geq (m+1)(n+1), \\ 1 & \text{if } \#S \leq (m+1)(n+1); \end{cases}$$

we then proceed exactly as in Lemma 7.7. This finishes the proof the this lemma, except in the following two cases.

Case 1: $m = 3$ and $(a, b, c) = (0, 0, 2)$. Write $S = \{q_1, q_2, \ldots, q_6, p_1, p_2, \ldots, p_{12}\}$. Let E_1 and E_2 be general elliptic normal curves. By specialization, we can suppose that

$$\begin{aligned} \{q_1, q_2, \ldots, q_6, p_1\} &\subset E_1 \cap E_2 \\ \{p_7, p_8, \ldots, p_{12}\} &\subset E_1 \smallsetminus E_2 \\ \{p_2, p_3, \ldots, p_6\} &\subset E_2 \smallsetminus E_1. \end{aligned}$$

We define $X = \{q_1, q_2, \ldots, q_6, p_1, p_7, p_8, \ldots, p_{12}\}$ and $Y = \{p_2, p_3, \ldots, p_6\}$. Now the exact sequence of sheaves

$$0 \to \mathcal{I}_{Y/Q}(1, 1) \to \mathcal{I}_{S/Q}(3, 3) \to \mathcal{I}_{X/E_1}(3, 3) \to 0$$

gives rise to a long exact sequence in cohomology

$$\cdots \to H^0(\mathcal{I}_{Y/Q}(1, 1)) \to H^0(\mathcal{I}_{S/Q}(3, 3)) \to H^0(\mathcal{I}_{X/E_1}(3, 3)) \to \cdots$$

As $H^0(\mathcal{I}_{Y/Q}(1, 1)) = H^0(\mathcal{I}_{X/E_1}(3, 3)) = 0$, we have $H^0(\mathcal{I}_{S/Q}(3, 3)) = 0$ as desired.

Case 2: $m = 4$ and $(a, b, c) = (0, 0, 3)$. Write $S = \{q_1, q_2, \ldots, q_6, p_1, p_2, \ldots, p_{18}\}$. Let E_1, E_2, and E_3 be three general elliptic normal curves on Q all passing through $\{q_1, q_2, \ldots, q_6\}$; let $C \subset Q$ be a general conic. By specialization, we can suppose that

$$\begin{aligned} \{q_1, q_2, \ldots, q_6\} &\subset E_1 \cap E_2 \cap E_3 \smallsetminus Q \\ \{p_1, p_2, \ldots, p_6\} &\subset E_1 \smallsetminus (E_2 \cup E_3 \cup Q) \\ \{p_7, p_8\} &\subset E_1 \cap E_2 \smallsetminus (E_3 \cup Q) \\ \{p_9, p_{10}, p_{11}, p_{12}\} &\subset E_2 \cap Q \smallsetminus (E_1 \cup E_3) \\ \{p_{13}, p_{14}\} &\subset E_1 \cap E_3 \smallsetminus (E_2 \cup Q) \\ \{p_{15}\} &\subset E_3 \cap Q \smallsetminus (E_1 \cup E_2) \\ \{p_{16}, p_{17}, p_{18}\} &\subset E_3 \smallsetminus (E_1 \cup E_2 \cup Q). \end{aligned}$$

We define

$$
\begin{aligned}
X &= \{q_1, q_2, \dots, q_6, p_1, p_2, \dots, p_8, p_{13}, p_{14}\} \\
Y &= \{p_9, p_{10}, p_{11}, p_{12}, p_{15}\} \\
Z &= \{p_{16}, p_{17}, p_{18}\}.
\end{aligned}
$$

As divisors on E_1, we have $X \sim 7H - 2\cdot\{q_1, q_2, \dots, q_6\}$, so in general, $X \not\sim 4H$. Consequently, we may assume $H^1(\mathcal{I}_{X/E_1}(4,4)) = 0$. To finish the proof, we consider the exact sequences of sheaves

$$
\begin{gathered}
0 \to \mathcal{I}_{(Y\cup Z)/Q}(2,2) \to \mathcal{I}_{S/Q}(4,4) \to \mathcal{I}_{X/E_1}(4,4) \to 0 \\
0 \to \mathcal{I}_{Z/Q}(1,1) \to \mathcal{I}_{(Y\cup Z)/Q}(2,2) \to \mathcal{I}_{Y/C}(2,2) \to 0.
\end{gathered}
$$

These give rise to the long exact sequences on cohomology

$$
\begin{gathered}
\cdots \to H^1(\mathcal{I}_{(Y\cup Z)/Q}(2,2)) \to H^1(\mathcal{I}_{S/Q}(4,4)) \to H^1(\mathcal{I}_{X/E_1}(4,4)) \to \cdots \\
\cdots \to H^1(\mathcal{I}_{Z/Q}(1,1)) \to H^1(\mathcal{I}_{(Y\cup Z)/Q}(2,2)) \to H^1(\mathcal{I}_{Y/C}(2,2)) \to \cdots
\end{gathered}
$$

As $H^1(\mathcal{I}_{Z/Q}(1,1)) = H^1(\mathcal{I}_{Y/C}(2,2)) = H^1(\mathcal{I}_{X/E_1}(4,4)) = 0$, we have $H^0(\mathcal{I}_{S/Q}(3,3)) = 0$ as desired. ☐

Theorem 1.4 now follows by combining Corollary 7.8, together with Lemma 7.12 to deal with the case $m = n$.

A Code for Lemma 7.11

In this section, we give python code to do the finite computations described in the proof of Lemma 7.11.

```
def plus(k, L):
   return L[:k] + [(L[k][0] - 1, L[k][1] - 1)] + L[k+1:]


def minus(k, L):
   return L[:k] + [(L[k][0] - 1, L[k][1] + 1)] + L[k+1:]


def check(k, L):
   return L[:k] + [(L[k][0] - 1, float('nan'))] + L[k+1:]


def doable(a, b, c, L):
   if (0, 0) in L:
      return False
```

```
    L = filter(lambda x : x[0] != 0, L)

    if L == []:
        return True

    if a != 0:
        for i in xrange(len(L)):
            if doable(a - 1, b, c, check(i, L)):
                return True
        return False

    if b != 0:
        for i in xrange(len(L)):
            if doable(a, b - 1, c, minus(i, L)):
                return True
        return False

    for i in xrange(len(L)):
        if L[i][0] >= 3:
            if doable(a, b, c - 1, minus(i, minus(i, plus(i, L)))):
                return True

    for i in xrange(len(L)):
        for j in xrange(i + 1, len(L)):
            if L[j][0] >= 2:
                if doable(a, b, c - 1, minus(i, minus(j, plus(j, L)))):
                    return True

    for i in xrange(len(L)):
        for j in xrange(i + 1, len(L)):
            for k in xrange(j + 1, len(L)):
                if doable(a, b, c - 1, minus(i, minus(j, plus(k, L)))):
                    return True

    return False


def verify(target):
    s = sum(target)
    L = [(target[0], -1)] + [(i, 0) for i in target[1:]]

    for c in xrange(1 + s/3):
        for b in xrange(1 + s - 3 * c):
            a = s - b - 3 * c
```

```
        if (a == 0) or (c == 0):
            if not doable(a, b, c, L):
                print a, b, c, ':', target


for m in xrange(7):
    print 'For m =', m, '...'

    mid = [2 * m - 4 * j for j in xrange(1, (m + 1)/2)]

    if m % 2 == 0:
        for n1 in xrange(2 * m - 3, 1 + (5 * m - 8) / 2):
            verify([n1] + mid + [1])
        for n1 in xrange((3 * m - 4) / 2, 1 + 2 * m - 3):
            verify([n1] + mid)

    else:
        for n1 in xrange((3 * m - 5) / 2, 1 + (5 * m - 7) / 2):
            verify([n1] + mid)

    if m == 5:
        for n1 in (6, 8, 10):
            verify([n1, 6, 2])

    if m == 6:
        for n1 in (8, 10, 12):
            for tail in ([], [1]):
                verify([n1, 8, 4] + tail)
```

The output is as follows:

```
For m = 0 ...
For m = 1 ...
For m = 2 ...
0 2 0 : [1, 1]
0 1 0 : [1]
For m = 3 ...
0 1 1 : [2, 2]
0 2 1 : [3, 2]
0 0 2 : [4, 2]
For m = 4 ...
0 0 3 : [5, 4]
For m = 5 ...
For m = 6 ...
```

References

[1] R. Hartshorne and A. Hirschowitz. Smoothing algebraic space curves. In *Algebraic geometry, Sitges (Barcelona), 1983*, volume 1124 of *Lecture Notes in Math.*, pages 98–131. Springer, Berlin, 1985.

[2] A. Hirschowitz. Sur la postulation générique des courbes rationnelles. *Acta Math.*, 146(3-4):209–230, 1981.