“All Concepts are Kan Extensions”: Kan Extensions as the Most Universal of the Universal Constructions

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by

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1 Introduction

When Samuel Eilenberg and Saunders MacLane first introduced the concept of a category in 1942, some mathematicians derided it as "general abstract nonsense" [7]. Categories seemed so theoretical that many doubted they would lend new insights in any field. Now, however, many category theorists view the "general abstract nonsense" phrase as a challenge, and have even co-opted the term for their own uses. They have demonstrated the tremendous power of and universal insights provided by category theory.

Category theory offers "a bird's eye view of mathematics. From high in the sky, details become invisible, but you can see patterns that were impossible to detect from ground level" [5]. Category theory provides a mechanism to describe similarities within and between different branches of mathematics. Advances or constructions in one branch can be translated into other branches. Category theory focuses on the abstract structure of objects rather than on the elements of those objects.

The most important notion in category theory is that of a universal construction. Universal constructions are defined by the unique existence of certain relationships; a universal construction is a construction that is initial in some category. This allows focus on the relationships themselves without ambiguity. There are many different manifestations of universality. Limits, adjoints, and representable functors all provide interconnected examples. This interconnectedness is not by chance. The universality in one construction is exactly what relates it to the other universal constructions.

The categorical notion of a Kan extension is a particular form of universal construction. Kan extensions offer an approach to extending one functor along another to create a third functor. Kan extensions, although "quite simple to define, [are] surprisingly ubiquitous throughout mathematics" [8].

In 1956 Henri Cartan and Eilenberg began using what would come to be known as Kan extensions to compute derived functors in homological algebra [2]. Two years later, in his paper "Adjoint Functors" [4], Daniel M. Kan computed some extensions, still not yet known as Kan extensions, through limits to calculate certain adjoint functors. By 1960, Kan had introduced the more general notion of these extensions, and they became known as Kan extension. By the time that MacLane published *Categories for the Working Mathematician* in 1978, category theory was both more developed and more popular and appreciated. MacLane's chapter on Kan extensions brought together some of the many uses and examples of Kan extensions, and brought the notion of Kan extensions into wider circulation.

Ending his chapter on Kan extensions, MacLane famously asserted that "the notion of Kan extensions subsumes all the other fundamental concepts of category theory." He shows how limits, adjoints and the Yoneda Lemma are just special Kan extensions. This thesis is an exposition of Kan extensions, drawing on and enlarging MacLane. In this, we hope to show that MacLane was not exaggerating in noting that "all concepts are Kan extensions." We argue that, in some ways, Kan extensions are the most universal of the universal constructions.

In Section 2 we define Kan extensions and give some basic examples. We will show how limits and colimits are special cases of Kan extensions, and how, when all the extensions exist, they define adjoint functors. In Section 3 we give limit and colimit formulae for Kan extensions, helping to find conditions for when Kan extensions will exist. In Section 4 we look at how Kan extensions act with other functors, defining the preservation of a Kan extension. This allows us to prove an adjoint functor theorem where the existence of an adjoint is equivalent to the existence and preservation of a certain Kan extension.

In Section 3, we saw that Kan extensions can be computed through limit and colimit formulae, provided the limits and colimits exist in the target category. We call these formulae "pointwise" since they define the Kan extensions object by object. In Section 5, we give an abstract definition of a "pointwise" Kan extension that we prove in Theorem 5.5 holds precisely when the limit and colimit formulae are in effect. This allows us to prove the Yoneda Lemma in a new way.

In Section 6, we relate the concept of density to Kan extensions, and prove how density theorems revolve around Kan extensions. This allows us to prove the co-Yoneda, or Density, Lemma in Corollary 6.4 with little effort. Finally, in Section 7, we reexamine some classic results from category theory and demonstrate how Kan extensions can ease the proofs of these important results.

A Note on Notation

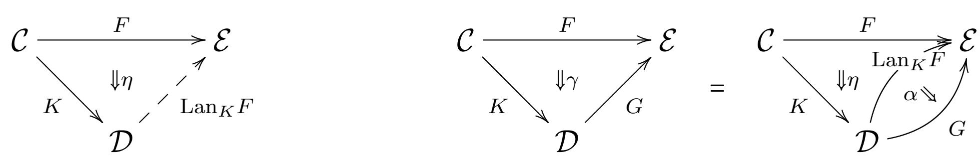
In this thesis we will use $\mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{F}$ as general categories and $\mathbf{1}$ as the terminal category. Categories in boldface, such as $\mathbf{Set}, \mathbf{Vect}_k$, or $\mathbf{Top}$, indicate the category of sets and set functions, or vector spaces and linear maps, or topological spaces and continuous maps respectively. The notation $[\mathcal{C}, \mathcal{D}]$ means the category of functors from $\mathcal{C}$ to $\mathcal{D}$. The notation $[\mathcal{C}, \mathcal{D}](F, G)$ then is the set of a natural transformations between F and G, functors with domain $\mathcal{C}$ and codomain $\mathcal{D}$. The reversed turnstile, $F \dashv G$ indicates that F is the left adjoint to G. In the diagrams, the arrows $\rightarrow$ indicate functors, while the double arrows $\Rightarrow$ indicate a natural transformation. Wherever possible, labels are above or below the arrows, however in some diagrams, there is simply not enough space, so the labels intersect the arrows.

2 Kan Extensions

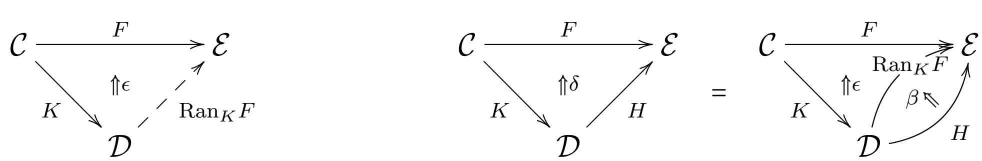
In this section, we introduce Kan extensions and give some basic examples. The first example shows how limits and colimits are just special cases of Kan extensions. Second, we show that when all the proper Kan extensions exist, they define functors that are left and right adjoints to the precomposition functor. Ultimately, what makes Kan extensions interesting and useful is the universality condition in the definition; this is what relates Kan extensions to adjoint functors and limits.

As the diagram in the definition below implies, a Kan extension of a functor $F{:}\mathcal{C} \to \mathcal{E}$ along $K{:}\mathcal{C} \to \mathcal{D}$ is an approximation of F which extends the domain of F from $\mathcal{C}$ to $\mathcal{D}$. Thus Kan extensions are simply a method for taking two functors and coming up with a third functor that attempts to make a certain triangle make sense. Requiring this triangle to be commutative is too restrictive, so instead Kan extensions rely on natural transformations to help make the extension be the best possible approximation of F along K.

Definition 2.1. Given functors $F{:}\mathcal{C} \to \mathcal{E}$ and $K{:}\mathcal{C} \to \mathcal{D}$ a *left Kan extension* of F along K is a functor $\mathrm{Lan}_K F{:}\mathcal{D} \to \mathcal{E}$ together with a natural transformation $\eta{:}F \Rightarrow \mathrm{Lan}_K F \circ K$ such that for any other such pair $(G{:}\mathcal{D} \to \mathcal{E}, \gamma{:}F \Rightarrow GK), \gamma$ factors uniquely through η. In other words, there exists a unique natural transformation $\alpha{:}\mathrm{Lan}_K F \Rightarrow G$ as illustrated.

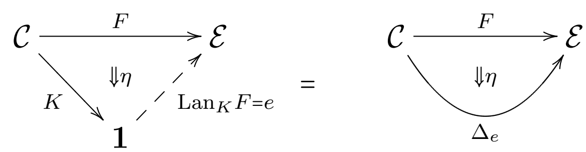


Definition 2.2. Dually, a *right Kan extension* of F along K is a functor $\mathrm{Ran}_K F{:}\mathcal{D} \to \mathcal{E}$ together with a natural transformation $\epsilon{:}\mathrm{Ran}_K F \circ K \Rightarrow F$ which is universal. That is, for any pair $(H{:}\mathcal{D} \to \mathcal{E}, \delta{:}H \circ K \Rightarrow F)$, there exists a unique factorization $\beta{:}H \Rightarrow \mathrm{Ran}_K F$ through ϵ as illustrated.



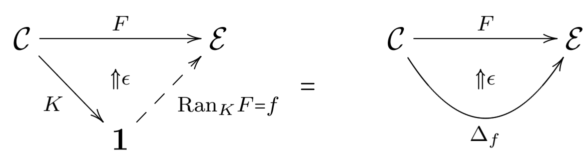
Remark 2.3. The general definition for both right and left Kan extensions does not refer to objects. Thus these definitions will work in *any* 2-category, not just the category of categories, **Cat**. However the diagrams and examples will tend to be from within **Cat**, and in later sections working within **Cat** becomes necessary.

Example 2.4. Let $\mathcal{D} = \mathbf{1}$ where $\mathbf{1}$ is the terminal category with one object 1 and one map 1_1. There is only one functor $K: \mathcal{C} \to \mathbf{1}$, the functor which maps all objects of $\mathcal{C}$ to the single object of $\mathbf{1}$ and all maps of $\mathcal{C}$ to the single map in $\mathbf{1}$. A left Kan extension of F along K is then a functor $\mathrm{Lan}_K F: \mathbf{1} \to \mathcal{E}$ together with a natural transformation $\eta: F \Rightarrow \mathrm{Lan}_K F \circ K$. However, a functor from the terminal category $\mathbf{1}$ to any category $\mathcal{E}$ is simply a choice of object e of $\mathcal{E}$, and precomposing that functor with the unique functor $K: \mathcal{C} \to \mathbf{1}$ is simply the constant functor $\Delta_e: \mathcal{C} \to \mathcal{E}$.



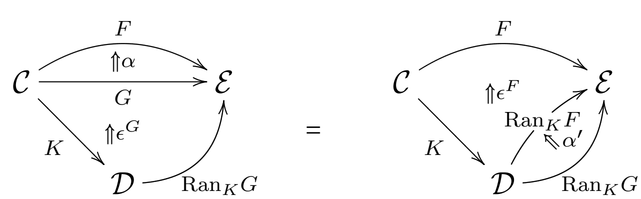
Thus the left Kan extension is a choice of e together with a natural transformation $\eta: F \Rightarrow \Delta_e$. This is simply a cone under F with vertex e. A natural transformation between two functors e and e' out of the terminal category is simply a map in $\mathcal{E}$ between e and e'. The universality of the left Kan extension says that any other cone under F factors uniquely through e. Thus $e = \mathrm{Lan}_K F(1)$ is the colimt of F.

Dually, a right Kan extension is a choice of object f in $\mathcal{E}$, and a natural transformation $\epsilon: \Delta_f \Rightarrow F$, or a cone over F, with the universal property that all other cones over F factor uniquely through f. Thus $f = \mathrm{Ran}_K F(1)$ is a limit of F.

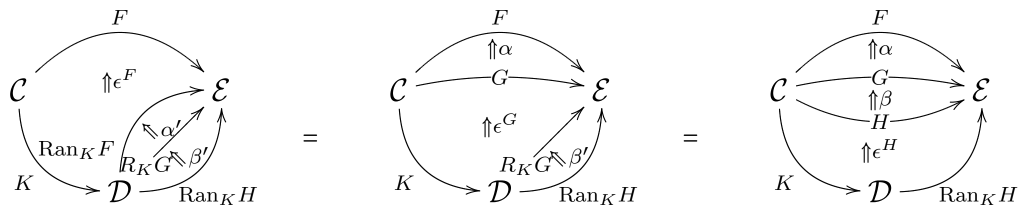


Remark 2.5. Kan extensions will not always exist. For example two objects could have empty hom sets in $\mathcal{C}$ and $\mathcal{E}$, but not in $\mathcal{D}$. For any K, there could be extensions for no, some or all functors F.

Now assume that for fixed K, both $\mathrm{Lan}_K F$ and $\mathrm{Ran}_K F$ exist for all F, with units η^F and counits ϵ^F respectively. Then, given $\alpha: G \Rightarrow F$, the pair $(\mathrm{Ran}_K G, \alpha \circ \epsilon^G)$ has a unique factorization through $(\mathrm{Ran}_K F, \epsilon^F)$, that is there exists a unique $\alpha': \mathrm{Ran}_K G \Rightarrow \mathrm{Ran}_K F$ such that $\alpha \circ \epsilon^G = \epsilon^F \circ \alpha'$.

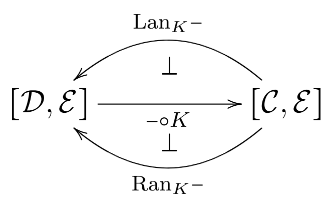


Thus for $F, G, H: \mathcal{C} \to \mathcal{E}$ with $\alpha: G \Rightarrow F$ and $\beta: H \Rightarrow G$,



So $\mathrm{Ran}_K(\alpha \circ \beta) = \alpha' \circ \beta'$, making composition follow functoriality conditions. Thus, for each fixed K we can define the functor $\mathrm{Ran}_K -: [\mathcal{C}, \mathcal{E}] \to [\mathcal{D}, \mathcal{E}]$ which takes a functor $F: \mathcal{C} \to \mathcal{E}$ to its right Kan extension along K. Similarly, for each K we have the functor $\mathrm{Lan}_K -: [\mathcal{C}, \mathcal{E}] \to [\mathcal{D}, \mathcal{E}]$ which takes a functor F to its left Kan extension along K.

Theorem 2.6. *If the Kan extensions exist for all F, then Lan_K- and Ran_K- are respectively the left and right adjoints to the functor $- \circ K$ which is precomposition with K.*



Proof. By the Yoneda Lemma, any pair (G, γ), as in the definition for the left Kan extension, yields a natural transformation

$$\gamma^*: [\mathcal{D}, \mathcal{E}](G, -) \Rightarrow [\mathcal{C}, \mathcal{E}](F, - \circ K)$$

by $\gamma^*_H(\alpha) = \alpha_K \circ \gamma$. The universal property of the left Kan extension says the natural transformation given by the pair $(\mathrm{Lan}_K F, \eta)$,

$$\eta^*: [\mathcal{D}, \mathcal{E}](\mathrm{Lan}_K F, -) \Rightarrow [\mathcal{C}, \mathcal{E}](F, - \circ K)$$

is a natural isomorphism. Thus $(\mathrm{Lan}_K F, \eta)$ represents the functor $[\mathcal{C}, \mathcal{E}](F, - \circ K)$. Dually, $(\mathrm{Ran}_K F, \epsilon)$ represents the functor $[\mathcal{C}, \mathcal{E}](- \circ K, F)$.

Since $(\mathrm{Lan}_K F, \eta)$ represents the functor $[\mathcal{C}, \mathcal{E}](F, - \circ K)$, for each F,

$$[\mathcal{D}, \mathcal{E}](\mathrm{Lan}_K F, G) \cong [\mathcal{C}, \mathcal{E}](F, G \circ K)$$

natural in G. This says precisely that $\mathrm{Lan}_K - \dashv - \circ K$. Similarly, since $(\mathrm{Ran}_K F, \epsilon)$ represents the functor $[\mathcal{C}, \mathcal{E}](- \circ K, F)$, for each F,

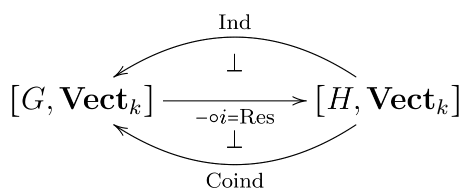
$$[\mathcal{D}, \mathcal{E}](G, \mathrm{Ran}_K F) \cong [\mathcal{C}, \mathcal{D}](G \circ K, F)$$

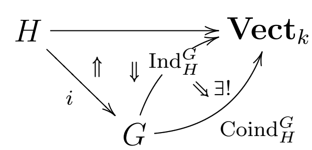
natural in G. Again, this is precisely the statement of $- \circ K \dashv \mathrm{Ran}_K -$.

Moreover, the 2-cells η define the components for the unit of the first adduction, while the 2-cells ϵ define the components of the counit for the second adjunction. □

Since adjoints are unique up to unique isomorphism, the left adjoint to any precomposition functor will be a left Kan extension functor, while the right adjoint to any precomposition functor will be a right Kan extension functor.

Example 2.7. A group can be thought of as a category which has only one object and all invertible maps. Let G be a group and H a subgroup of G with the inclusion functor $i: H \to G$. The category $[G, \mathbf{Vect}_k]$ of functors from the group G to the category of vector spaces has objects which are G−representations of over a fixed field k and has arrows which are G−equivariant linear maps. Since any representation of G can be restricted through the inclusion map i to a representation of H, there is a functor $\mathrm{Res}: [G, \mathbf{Vect}_k] \to [H, \mathbf{Vect}_K]$ defined by precomposition with i.





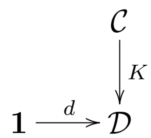
Moreover, any representation on H can be extended to the induced representation of G, by induction, or extended to the coinduced representation of G, by coinduction. These extensions define functors, $\mathrm{Ind}, \mathrm{Coind}: [H, \mathbf{Vect}_k] \to [G, \mathbf{Vect}_k]$, which are left and right adjoints to the restriction functor. Since these functors are adjoint to a precomposition functor, they define left and right Kan extensions along the inclusion functor.

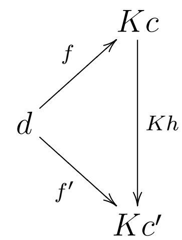
3 Limit and Colimit Formulae

In the last section, we saw that a right Kan extension through the terminal category was a limit. Interestingly, this relationship holds in the other direction - that is, not only are right Kan extensions limits, but limits can also define right Kan extensions.

In this section, we will show how right Kan extensions can be calculated at each object by taking limits. Dually, we show left Kan extensions can be computed through pointwise colimits. This gives easy conditions for when left and right Kan extensions must exist, as well as when the natural transformation of the extension will be a natural isomorphism.

To use a limit formula to calculate the Kan extensions, not surprisingly, requires that certain limits exist. Specifically, we need the limits over the comma category $(d \downarrow K)$ to exist for each $d \in \mathcal{D}$. Note that to create these comma categories now requires working in **Cat** rather than an arbitrary 2-category, since we refer to objects and maps within the category $\mathcal{D}$.





The comma category $(d \downarrow K)$ has objects which are pairs $(f\colon d \to Kc, c \in \mathcal{C})$, and maps $h\colon (f, c) \to (f', c')$ which are maps $h\colon c \to c'$ in $\mathcal{C}$ such that the triangle $Kh \circ f = f'$ commutes. Along with each comma category there is a projection functor $Q^d\colon (d \downarrow K) \to \mathcal{C}$ which maps (f, c) to c.

When the limit of the composite functor FQ^d exists for each d, not only does the right Kan extension exist, but the functor is given by taking the limit at each object and map of $\mathcal{D}$, and the natural transformation has components from each of the limiting cones over FQ^d. The following theorem and proof appear similarly as MacLane's Theorem X.3.1 in [6].

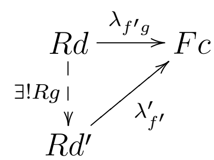
Theorem 3.1. *Given $K\colon \mathcal{C} \to \mathcal{D}$, let $F\colon \mathcal{C} \to \mathcal{E}$ be a functor such that the composite $(d \downarrow K) \xrightarrow{Q^d} \mathcal{C} \xrightarrow{F} \mathcal{E}$ has for each $d \in \mathcal{D}$ a limit in $\mathcal{E}$, with limiting cone λ, written*

$$Rd = \mathrm{Lim}((d \downarrow K) \xrightarrow{Q^d} \mathcal{C} \xrightarrow{F} \mathcal{E}) \tag{3.2}$$

Then each $g\colon d \to d'$ induces a unique arrow $Rg\colon \mathrm{Lim}FQ^d \to \mathrm{Lim}FQ^{d'}$, so R is a functor. Moreover, for each $c \in \mathcal{C}$, the components $\lambda_{1_{Kc}} = \epsilon_c$ define a natural transformation, and (R, ϵ) is the right Kan extension of F along K.

Proof. First, we want to show R as defined is indeed a functor; second, that ϵ as defined is indeed natural; and finally, that the pair (R, ϵ) is universal, and thus the right Kan extension.

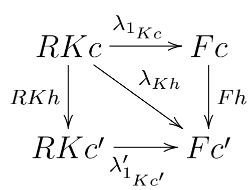
Limits of FQ^d are functors out of $(d \downarrow K)$, and thus functors of d. Specifically, given some $g\colon d \to d'$ and $Q^{d'}\colon (d' \downarrow K) \to \mathcal{C}$, each $f'\colon d' \to Kc$ determines $f'g \in (d \downarrow K)$. Then we have limit cones over FQ^d and $FQ^{d'}$, with $\lambda\colon \Delta_{Rd} \to FQ^d$ and $\lambda'\colon \Delta_{Rd'} \to FQ^{d'}$. The components $\lambda_{f'g}\colon Rd \to Fc$ form a cone with vertex Rd. However, the components $\lambda_{f'}\colon Rd' \to Fc$ form a cone with vertex Rd'. Since λ' was a limiting cone, there exists a unique arrow, $Rg\colon Rd \to Rd'$ such that



(3.3)

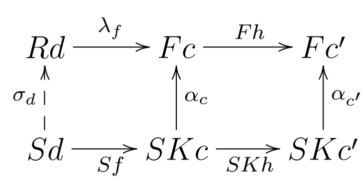
commutes for all $f' \in (d' \downarrow K)$. This makes R a functor from $\mathcal{D}$ to $\mathcal{E}$, as desired.

For each $c \in \mathcal{C}$ the identity map $1_{K_c} \in (Kc \downarrow K)$, so the limiting cone λ has a component $\lambda_{1_{Kc}}\colon RKc \to Fc$, called ϵ_c. Let $h\colon c \to c'$ be a map in $\mathcal{C}$. Then



commutes. The top triangle commutes because λ is a cone. The bottom triangle commutes by definition of Rg as shown in (3.3); specifically, let $Kh = g$. The top arrow $\lambda_{1_{Kc}}$ is precisely ϵ_c, while the bottom arrow $\lambda'_{1_{Kc'}}$ is $\epsilon_{c'}$. Thus the outer square is exactly a naturality square for ϵ, so $\epsilon\colon RK \Rightarrow F$ is a natural transformation, as desired.

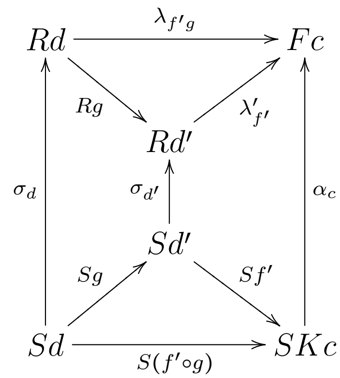
Let $S\colon \mathcal{D} \to \mathcal{E}$ be a functor with $\alpha\colon SK \Rightarrow F$ natural. Define $\sigma\colon S \to R$ by component as the unique arrow which makes diagram (3.4) commute.



(3.4)

The right square commutes by the naturality of α, and the composite maps $\alpha_c \circ Sf$ form a cone with summit Sd. However, λ is a limit cone, so there exists some unique arrow $\sigma_d\colon Sd \to Rd$ to make the left-hand square, and thus the whole diagram, commute. Now that σ is defined on components, we want to show that it defines a natural transformation.

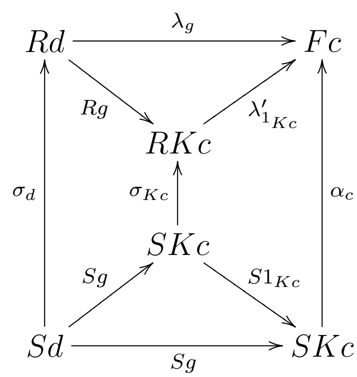
In diagram (3.5), the outer rectangle and the right trapezoid commute by definition of σ_d and $\sigma_{d'}$. The top triangle commutes by definition of Rg, while the bottom triangle commutes by functoriality of S.



(3.5)

This yields $\lambda'_{f'} \circ Rg \circ \sigma_d = \lambda'_{f'} \circ \sigma_{d'} \circ Sg$ for all $f' \in (d' \downarrow K)$, and $g \in \mathcal{D}$. However, since λ' was a limit cone, this implies $Rg \circ \sigma_d = \sigma_{d'} \circ Sg$, so σ is indeed natural.

Let $d' = Kc$, $f' = 1_{Kc}$, and $c = c'$. Then the diagram (3.5) becomes



(3.6)

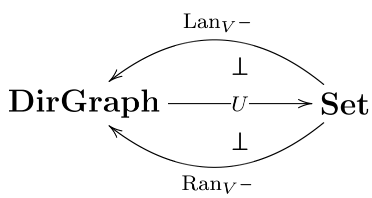
which holds for every c. Thus from the right trapezoid in diagram (3.6), $\alpha = \epsilon \circ \sigma K$, or α factors through ϵ, as desired. From the outer rectangle, we get $\lambda_g \circ \sigma_d = \alpha_c \circ Sg$. However, λ was a limit cone and thus universal, so σ is completely determined by α and S, and therefore is unique. This means that the pair (S, α) factors uniquely through (R, ϵ), so (R, ϵ) is the right Kan extension of F along K, as desired. □

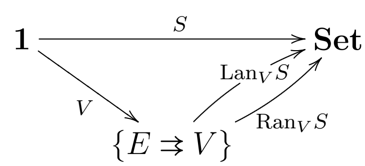
Dually, left Kan extensions can be computed pointwise through colimits, if all the proper colimits exist. Then,

$$\mathrm{Lan}_K F(d) = \mathrm{Colim}\big((K \downarrow d) \xrightarrow{\bar{Q}^d} \mathcal{C} \xrightarrow{F} \mathcal{E}\big) \tag{3.7}$$

That is, $\mathrm{Lan}_K F$ is computed pointwise by taking the colimit of F composed with the projection out of $(K \downarrow d)$, where $(K \downarrow d)$ is the comma category with objects being pairs $(c \in \mathcal{C}, f\colon Kc \to d)$, and the projection map $\bar{Q}^d\colon (K \downarrow d) \to \mathcal{C}$ maps (c, f) to c.

Example 3.8. Directed graphs are objects of the functor category $[\{E \rightrightarrows V\}, \mathbf{Set}]$, where the category $\{E \rightrightarrows V\}$ is the category with two objects, E and V, and four maps, $1_E, 1_V, s, t$. There is a forgetful functor $U\colon [\{E \rightrightarrows V\}, \mathbf{Set}] \to \mathbf{Set}$ which takes a graph to its set of vertices. This forgetful functor is given by restricting along the functor from the terminal category $\mathbf{1}$. Since $[\mathbf{1}, \mathbf{Set}] = \mathbf{Set}$, we can compute a left and right adjoint to the forgetful functor by using the formulas of Theorem 3.1 and its dual, equation (3.7).





The colimit formula of equation (3.7) allows us to calculate the left Kan extension at the objects V and E. Thus

$$(\mathrm{Lan}_V S)V = \mathrm{Colim}((V \downarrow V) \xrightarrow{\bar{Q}^E} \mathbf{1} \xrightarrow{S} \mathbf{Set}) = \coprod_{\mathbf{1}} S = S$$

Here the comma category $(V \downarrow V)$ is equal to the terminal category, since it has one element 1_V, so the coproduct of S indexed by $\mathbf{1}$ is simply the set S. To calculate the extension at E,

$$(\mathrm{Lan}_V S)E = \mathrm{Colim}((V \downarrow E) \xrightarrow{\bar{Q}^E} \mathbf{1} \xrightarrow{S} \mathbf{Set}) = \varnothing$$

since the comma category $(V \downarrow E)$ is empty. Thus, the left adjoint to the forgetful functor takes a set and maps it to the graph which has vertices of the set, but no edges.

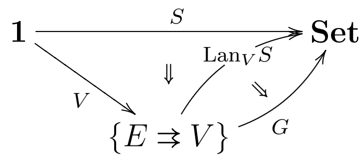
Dually, the limit formula of Theorem 3.1 gives the right Kan extension at the object V as

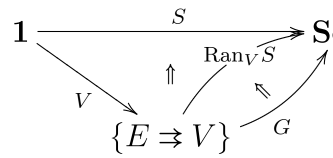
$$(\mathrm{Ran}_V S)V = \mathrm{Lim}((V \downarrow V) \xrightarrow{Q^E} \mathbf{1} \xrightarrow{S} \mathbf{Set}) = \prod_{\mathbf{1}} S = S$$

Again the comma category $(V \downarrow V) = \mathbf{1}$, so the product over $\mathbf{1}$ is simply S. At the object E,

$$(\mathrm{Ran}_V S)E = \mathrm{Lim}((E \downarrow V) \xrightarrow{Q^E} \mathbf{1} \xrightarrow{S} \mathbf{Set}) = \prod_{(s,t)} S = S \times S$$

since the comma category $(E \downarrow V)$ has two disconnected objects, s and t. Thus, the right adjoint to U takes a set S and maps it to the chaotic graph which has vertices of the set, and two edges between every vertex, one in each direction.



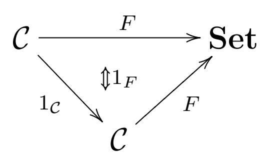


If G is a graph with vertices v and edges e, then a map of sets $S \to v$ is a natural transformation $S \Rightarrow GV$, and this is the same thing as a map of graphs $\mathrm{Lan}_V S \Rightarrow G$ because $\mathrm{Lan}_V S$ is a discrete. Similarly, a set map $v \to S$ is the same as a map of graphs $G \Rightarrow \mathrm{Ran}_V S$ because $\mathrm{Ran}_V S$ defines the chaotic graph which has all possible edges. Thus the Kan extensions calculated pointwise by limits and colimits do indeed define adjoints to the forgetful functor.

Corollary 3.9. *When $\mathcal{C}$ is small, right Kan extensions exist when $\mathcal{E}$ is complete, while left Kan extensions exist when $\mathcal{E}$ is cocomplete.*

Proof. When $\mathcal{C}$ is small and $\mathcal{E}$ complete, then all the proper limits of $(d \downarrow K) \xrightarrow{Q^d} \mathcal{C} \xrightarrow{F} \mathcal{E}$ exist, so the right Kan extension can be computed pointwise through these limits. Dually, when $\mathcal{C}$ small and $\mathcal{E}$ cocomplete, all the colimits of $(K \downarrow d) \xrightarrow{\bar{Q}^d} \mathcal{C} \xrightarrow{F} \mathcal{E}$ exist, so the left Kan extension can be computed pointwise through these colimits. □

Example 3.10. When $\mathcal{E} = \mathbf{Set}$, then $\mathcal{E}$ is both complete and cocomplete. When $\mathcal{C}$ is small, all functors $F: \mathcal{C} \to \mathbf{Set}$ have both right and left Kan extensions along all functors $K: \mathcal{C} \to \mathcal{D}$. This is the case that was originally studied by Kan in his paper on adjoint functors where he introduces the Kan extension as a pointwise limit [4].



Specifically, any functor $F: \mathcal{C} \to \mathbf{Set}$ has both right and left Kan extensions along $1_{\mathcal{C}}: \mathcal{C} \to \mathcal{C}$. The universal property of both right and left Kan extensions says that both $\mathrm{Ran}_I F$ and $\mathrm{Lan}_I F$ will be simply F itself, with the identity transformation $1_F: F \Rightarrow F$ being equal to both η and ϵ.

Corollary 3.11. *When K is fully faithful, $\epsilon: \mathrm{Ran}_K F \circ K \Rightarrow F$ is an isomorphism.*

Proof. When K is fully faithful, every $f\colon Kc \to Kc'$ can be written as $f = Kh$ for some unique $h\colon c \to c'$ in $\mathcal{C}$. Thus the object $(1_{Kc}, Kc)$ is initial in the comma category $(Kc \downarrow K)$, so the limit can be found by evaluating at the initial object. Thus

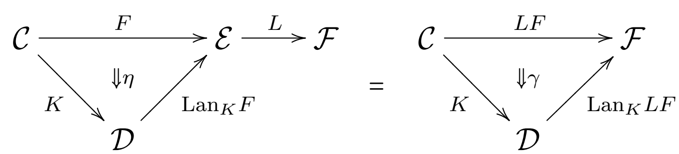
$$(\mathrm{Ran}_K F)Kc = \mathrm{Lim}((Kc \downarrow K) \xrightarrow{Q^{Kc}} \mathcal{C} \xrightarrow{F} \mathcal{E}) = Fc$$

This makes $\epsilon_c = 1$, for all $c \in \mathcal{C}$, so ϵ is an isomorphism. □

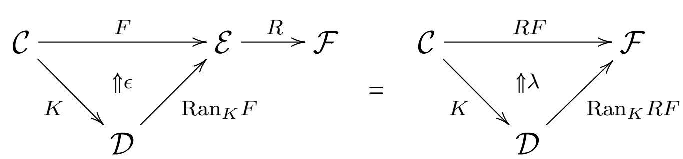
4 Preserving Extensions

How do Kan extensions interact with other functors, specifically when composed with other functors? In this section we look at functors which are said to *preserve* Kan extensions. A functor preserves a Kan extension when composing then extending is equivalent to extending then composing. We show that left adjoints preserve left Kan extensions, while right adjoints will preserve right adjoints. These connections with adjoints run deeper. We will show an adjoint functor theorem which says the existence of an adjoint is conditional on a functor having and preserving certain Kan extensions.

Definition 4.1. A functor $L\colon \mathcal{E} \to \mathcal{F}$ *preserves* a left Kan extension $(\mathrm{Lan}_K F, \eta)$ of F along K if $L \circ \mathrm{Lan}_K F$ is a left Kan extension of LF along K with counit $L\eta$.



Dually, a functor $R\colon \mathcal{E} \to \mathcal{F}$ *preserves* a right Kan extension $(\mathrm{Ran}_K F, \epsilon)$ of F along K if $R \circ \mathrm{Ran}_K F$ is a right Kan extension of RF along K with unit $R\epsilon$.



The following theorem and proof closely follows the arguments of Lemma 1.3.3 in Riehl [8].

Theorem 4.2. *Left adjoints preserve left Kan extensions.*

Proof. Let $L: \mathcal{E} \to \mathcal{F}$ be left adjoint to $R: \mathcal{F} \to \mathcal{E}$, with unit $\iota: 1_{\mathcal{E}} \Rightarrow R \circ L$ and counit $\nu: L \circ R \Rightarrow 1_{\mathcal{F}}$ This adjunction gives, for each $H: \mathcal{D} \to \mathcal{F}$ and each $G: \mathcal{D} \to \mathcal{E}$ the isomorphism

$$[\mathcal{D}, \mathcal{F}](LG, H) \cong [\mathcal{D}, \mathcal{E}](G, RH)$$

Thus letting G be the left Kan extension of F along K,

$$\begin{aligned} [\mathcal{D}, \mathcal{F}](L\mathrm{Lan}_K F, H) &\cong [\mathcal{D}, \mathcal{E}](\mathrm{Lan}_K F, RH) \\ &\cong [\mathcal{C}, \mathcal{E}](F, RHK) \\ &\cong [\mathcal{C}, \mathcal{F}](LF, HK) \end{aligned}$$

However, this isomorphism is simply the universal property of left Kan extensions, so $L\mathrm{Lan}_K F = \mathrm{Lan}_K LF$.

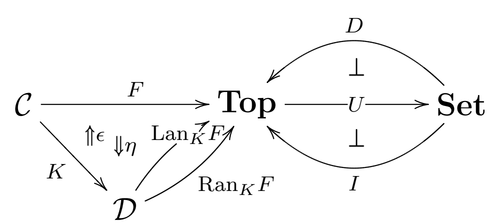
To find the unit of the extension, let $H = L\mathrm{Lan}_K F$ and follow $1_{L\mathrm{Lan}_K F}$ through the chain of isomorphisms, as shown:

$$1_{L\mathrm{Lan}_K F} \mapsto \iota_{\mathrm{Lan}_K F} \mapsto \iota_{\mathrm{Lan}_K F \circ K} \circ \eta \mapsto \nu_{L\mathrm{Lan}_K F \circ K} \circ L\iota_{\mathrm{Lan}_K F \circ K} \circ L\eta$$

By one of the triangle identities of the adjunction $L \dashv R$, the last element in the chain of isomorphisms simplifies, with $\nu_{L\mathrm{Lan}_K F \circ K} \circ L\iota_{\mathrm{Lan}_K F \circ K} \circ L\eta = L\eta$. Thus $(L\mathrm{Lan}_K F, L\eta)$ is the left Kan extension of LF along K, as desired. □

Dually, right adjoints preserve right Kan extensions.

Example 4.3. The forgetful functor $U: \mathbf{Top} \to \mathbf{Set}$ has both left and right adjoints. The functor $D: \mathbf{Set} \to \mathbf{Top}$, which assigns a set the discrete topology is left adjoint to U, while the functor $I: \mathbf{Set} \to \mathbf{Top}$, which assigns a set the indiscreet topology, is right adjoint to U. Thus the functor U preserves both left and right Kan extensions.



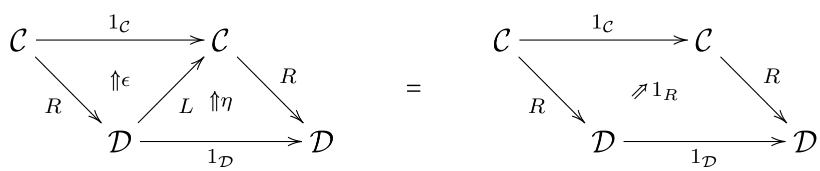
Corollary 4.4. *If* $(\mathrm{Ran}_K F, \epsilon)$ *is a right Kan extension and* $\mathcal{E}$ *is locally small and has all small copowers, then for each* $e \in \mathcal{E}$ *the functor* $\mathcal{E}(e, \mathrm{Ran}_K F-): \mathcal{D} \to \mathbf{Set}$ *is the right Kan extension of* $\mathcal{E}(e, F-): \mathcal{C} \to \mathbf{Set}$*, with counit* $\mathcal{E}(e, \epsilon): \mathcal{E}(e, \mathrm{Ran}_K F \circ K-) \Rightarrow \mathcal{E}(e, F-)$.

Proof. The functor $\mathcal{E}(e,-)\colon\mathcal{E}\to\mathbf{Set}$ is the right adjoint of the copower functor, the functor which maps x to $\coprod_x e$ □

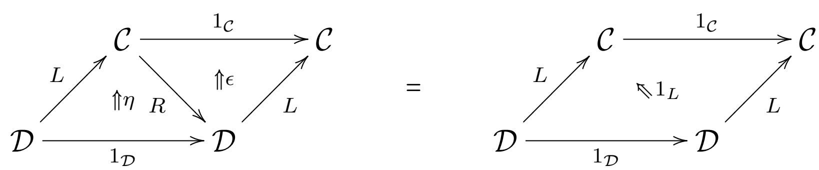
Both MacLane [6] and Borceux [1] prove the following adjoint functor theorem. Here, we give a 2-categorical reinterpretation of the standard proof.

Theorem 4.5. *A functor $R\colon\mathcal{C}\to\mathcal{D}$ has a left adjoint if and only if the right Kan extension of $1_\mathcal{C}$ along R exists and is preserved by R. When this is the case, $\mathrm{Ran}_R 1_\mathcal{C}\dashv R$ and $\epsilon\colon\mathrm{Ran}_R 1_\mathcal{C}\circ R\Rightarrow 1_\mathcal{C}$ for the Kan extension is the counit of the adjunction.*

Proof. Assume $L\dashv R$ with unit $\eta\colon 1_\mathcal{D}\Rightarrow RL$ and counit $\epsilon\colon LR\Rightarrow 1_\mathcal{C}$. The the triangle identities are given by

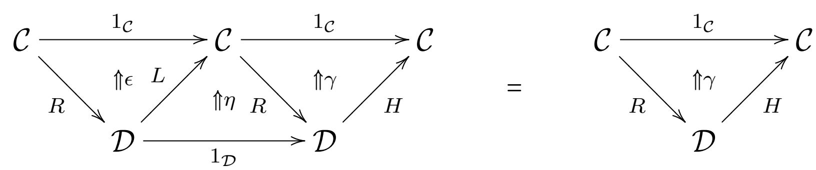


(4.6)

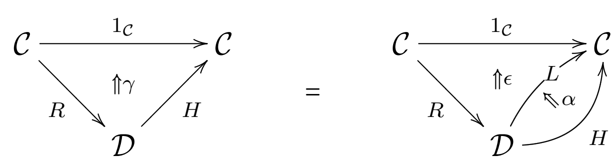


(4.7)

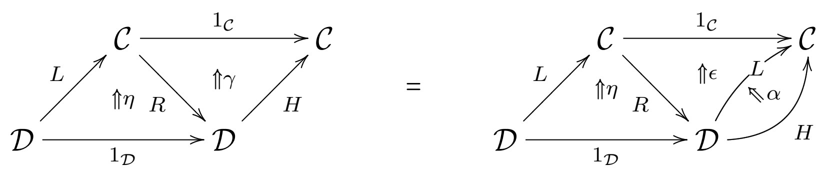
Then for all pairs $(H\colon\mathcal{C}\to\mathcal{C},\gamma\colon HR\Rightarrow 1_\mathcal{C})$, the following diagrams are equal thanks to the first triangle identity (4.6).



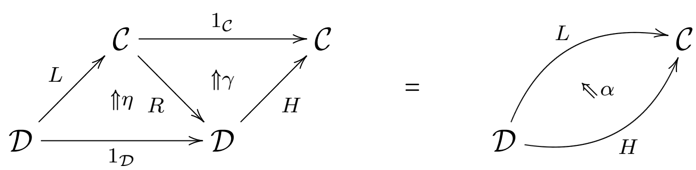
This shows the existence of a factorization of (H,γ) through the pair (L,ϵ). To show the factorization is unique, let $\alpha\colon H\Rightarrow L$ be a factorization of γ. Then



Adding the η 2-cell on to both sides yields

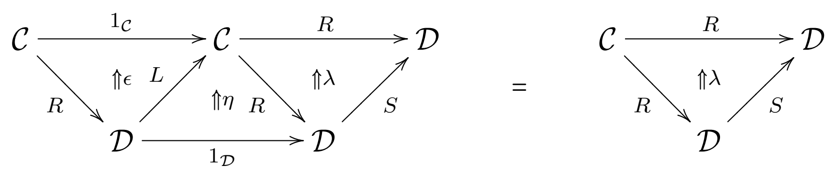


The right hand side simplifies by the second triangle identity (4.7), giving

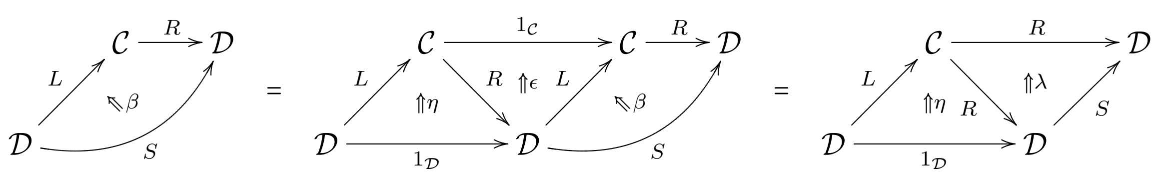


which shows the factorization through η is unique. Thus, the pair (L, ϵ) is the right Kan extension of $1_{\mathcal{C}}$ along R.

To show R preserves this extension, consider a pair $(S: \mathcal{D} \to \mathcal{D}, \lambda: SR \Rightarrow R)$. Using the same method as above, we can use the first triangle identity (4.6) to show that λ factors through $R\epsilon$.

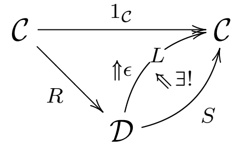


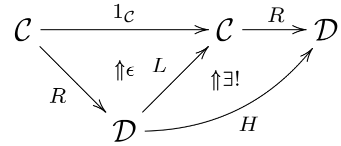
And again, we can use the second triangle identity (4.7) to show this factorization is unique. Let $\beta: S \Rightarrow LR$ be a factorization of λ through $R\epsilon$, then



where the left equality holds thanks to the second triangle identity (4.7) and the right equality holds since β was defined as a factorization of λ through $R\epsilon$. Since factorization is unique, the pair $(RL, R\epsilon)$ is the right Kan extension of R along R; in other words, R preserves the extension L. Thus if $L \dashv R$, $L = \mathrm{Ran}_R 1_{\mathcal{C}}$ and R preserves this extension, as desired.

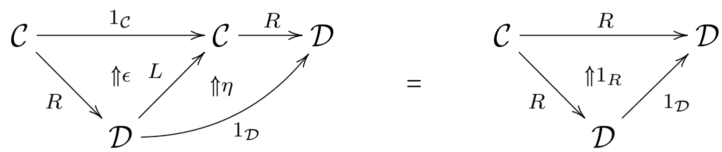
Now assume $L = \mathrm{Ran}_R 1_{\mathcal{C}}$ and $RL = \mathrm{Ran}_R R$ with counts of ϵ and $R\epsilon$ respectively.



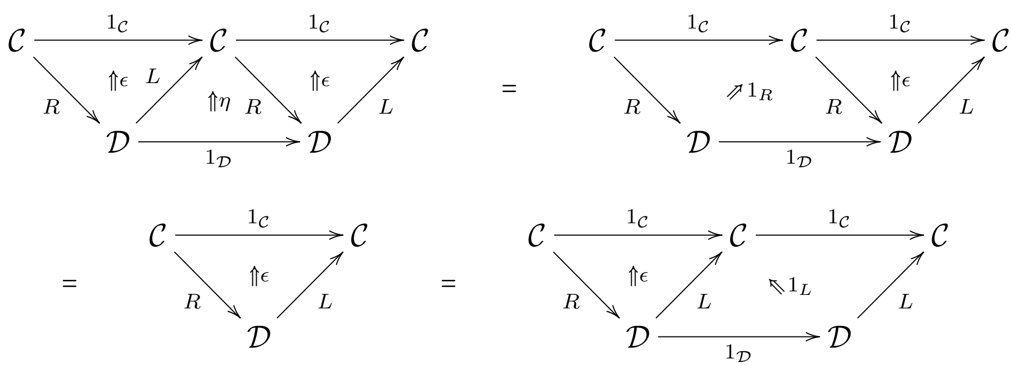


The universal properties of $\mathrm{Ran}_R 1_{\mathcal{C}}$ and $\mathrm{Ran}_R R$ yield unique factorizations for every S and H.

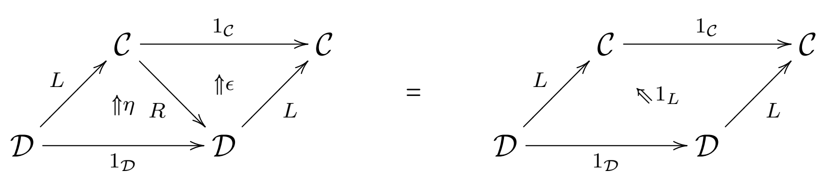
Define η to be the unique factorization from the pair $(1_{\mathcal{D}}, 1_R\colon R \Rightarrow R)$ for the extension $RL = \mathrm{Ran}_R R$, that is the unique natural transformation $1_{\mathcal{D}} \Rightarrow RL$ such that the following equality holds



This definition defines both a unit and a counit of an adjunction, and shows one triangle identity (4.6). To show the other triangle identity, consider



The first equality holds by our definition of η, the second by reducing the identity two cell 1_R and the third by adding the identity two cell 1_L. Since L was a right Kan extension, the factorization from L to L must be unique, so

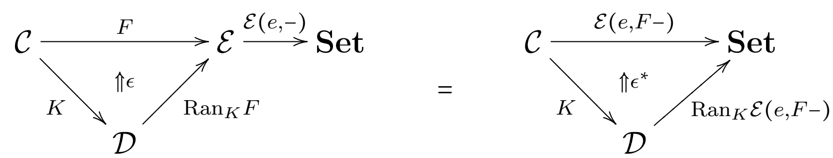


Which is the other triangle identity (4.7). Thus $L = \mathrm{Ran}_R 1_{\mathcal{C}}$ and $RL = \mathrm{Ran}_R R$ gives $L \dashv R$ as desired. $\square$

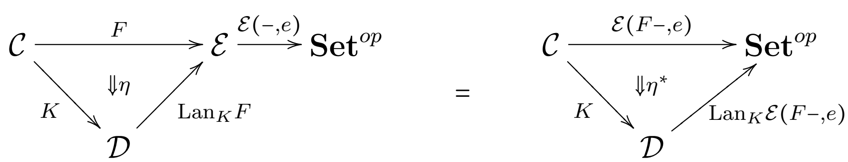
5 Pointwise Kan Extensions

In this section, we introduce a new definition of a pointwise Kan extension. Although at first glance this definition of a pointwise right Kan extension appears only distantly related to our previous notion of "pointwise" defined extensions through limits, we will show the two notions of pointwise are equivalent. This will allow a very interesting example in which we return to the simple case of Kan extensions into **Set**, and end up proving the Yoneda Lemma in a very different manner than the usual proof. Finally, using the duality of **Cat** and $\mathbf{Cat}^{co}$, we will show that two notions of pointwise are also equivalent in the case of left Kan extensions, better motivating our slightly peculiar definition of a pointwise left Kan extension.

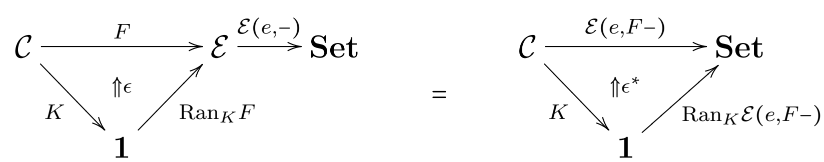
Definition 5.1. If $\mathcal{E}$ has small hom sets, a right Kan extension of F along K is *pointwise* if it is preserved by representable functors $\mathcal{E}(e,-): \mathcal{E} \to \mathbf{Set}$ for all $e \in \mathcal{E}$.



Definition 5.2. If $\mathcal{E}$ has small hom sets, a left Kan extension of F along K is *pointwise* if it is preserved by functors $\mathcal{E}(-,e): \mathcal{E} \to \mathbf{Set}^{op}$ for all $e \in \mathcal{E}$.

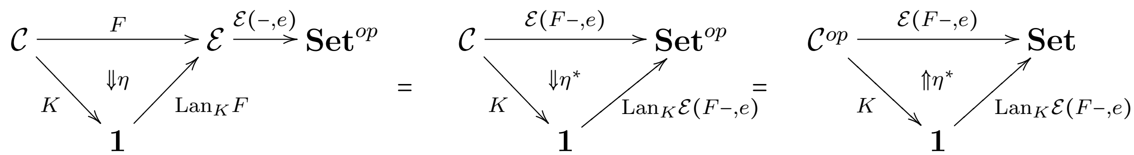


Example 5.3. Consider a right Kan extension of some functor $F: \mathcal{C} \to \mathcal{E}$ along the unique functor $K: \mathcal{C} \to \mathbf{1}$. Example 2.4 showed $\mathrm{Ran}_K F$ was simply the limit of F. Since representable functors preserve limits, $\mathcal{E}(e, \mathrm{Lim}F) = \mathrm{Lim}(\mathcal{E}(e, F-))$, with the canonical universal cone. Put another way,



Thus representables preserving limits means any right Kan extension through the terminal category is a pointwise right Kan extension.

Dually, consider a left Kan extension through the terminal category. In Example 2.4 we also showed $\mathrm{Lan}_K F$ was a colimit of F. Since representable functors reverse colimits, $\mathcal{E}(\mathrm{Lan}_K F, e) = \mathcal{E}(\mathrm{Colim} F, e) = \mathrm{Lim}(\mathcal{E}(F-, e))$ where $\mathcal{E}(F-, e): \mathcal{C} \to \mathbf{Set}^{op}$, again with the canonical universal cone. Thus,



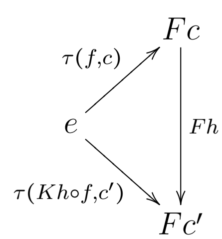
Note that η^* has components in $\mathbf{Set}^{op}$, so $\eta^*_c: \mathrm{Lim}(\mathcal{E}(F-, e)) \to \mathcal{E}(Fc, e)$, and thus η^* indeed acts as a limiting cone for $\mathcal{E}(F-, e)$, rather than a colimit cone as it might first appear. Thus any left Kan extension through the terminal category is a pointwise left Kan extension.

Lemma 5.4. *Given functors $K: \mathcal{C} \to \mathcal{D}$ and $F: \mathcal{C} \to \mathcal{E}$, there is a bijection*

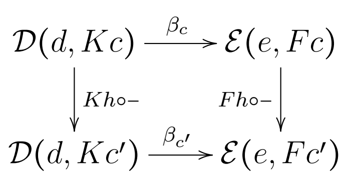
$$[(d \downarrow K), \mathcal{E}](\Delta_e, FQ^d) \cong [\mathcal{C}, \mathbf{Set}](\mathcal{D}(d, K-), \mathcal{E}(e, F-))$$

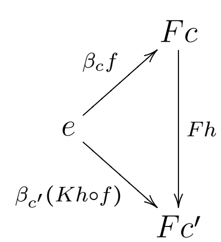
where the left hand side is equivalent to the cones over FQ^d with vertex e.

Proof. A natural transformation $\tau: \Delta_e \Rightarrow FQ^d$ assigns to each $f: d \to Kc \in (d \downarrow K)$ and each $c \in \mathcal{C}$ an arrow $\tau(f, c): e \to Fc$ subject to the condition that for each $h: c \to c'$, the following triangle commutes.



However, each natural transformation $\beta: \mathcal{D}(d, K-) \Rightarrow \mathcal{E}(e, F-)$ assigns to each $c \in C$ and each $f: d \to Kc \in (d \downarrow K)$ an arrow $\beta_c f: e \to Fc$ subject to the naturality condition that for each $h: c \to c'$, the square below commutes.





However, the requirement that the square commutes is equivalent to the requirement that for each $f \in (d \downarrow K)$, the triangle above commutes. Thus the bijection between $[(d \downarrow K), \mathcal{E}](\Delta_e, FQ^d)$ and $[\mathcal{C}, \mathbf{Set}](\mathcal{D}(d, K-), \mathcal{E}(e, F-))$ becomes clear, as $\tau(f, c)$ maps to $\beta_c f$, and vice versa. □

A similar proof to the following theorem, and the previous lemma, appear as Theorem X.5.3 in MacLane [6].

Theorem 5.5. *A functor $F: \mathcal{C} \to \mathcal{E}$ has a pointwise right Kan extension if and only if $(d \downarrow K) \xrightarrow{Q^d} \mathcal{C} \xrightarrow{F} \mathcal{E}$ has a limit for all d. Then $\mathrm{Ran}_K F$ is given by the pointwise limit formula of Theorem 3.1.*

Proof. Since representable functors preserve limits, any right Kan extension given by the limit formula will be pointwise. Thus one direction is obvious.

Conversely, assume for each $e \in \mathcal{E}$ the functor $\mathcal{E}(e, F-): \mathcal{C} \to \mathbf{Set}$ has a right Kan extension along K, $\mathcal{E}(e, \mathrm{Ran}_K F-): \mathcal{D} \to \mathbf{Set}$. Then the Yoneda lemma gives

$$\mathcal{E}(e, \mathrm{Ran}_K Fd) \cong [\mathcal{D}, \mathbf{Set}](\mathcal{D}(d, -), \mathcal{E}(e, \mathrm{Ran}_K F-))$$

By the defining universal property of Kan extensions,

$$[\mathcal{D}, \mathbf{Set}](\mathcal{D}(d, -), \mathcal{E}(e, \mathrm{Ran}_K F-)) \cong [\mathcal{C}, \mathbf{Set}](\mathcal{D}(d, K-), \mathcal{E}(e, F-))$$

Finally, by Lemma 5.4,

$$[\mathcal{C}, \mathbf{Set}](\mathcal{D}(d, K-), \mathcal{E}(e, F-)) \cong [(d \downarrow K), \mathcal{E}](\Delta_e, FQ^d)$$

Thus,

$$\mathcal{E}(e, \mathrm{Ran}_K Fd) \cong [(d \downarrow K), \mathcal{E}](\Delta_e, FQ^d)$$

But this isomorphism simply says that the set of cones over FQ^d is represented by $\mathrm{Ran}_K Fd$, so the limit of FQ^d exists, and thus $\mathrm{Ran}_K F$ can be computed through the pointwise limit formula of Theorem 3.1. □

While MacLane and others show the result of the following example using ends and coends, we have avoided introducing the calculus of ends by finding the same result in the pointwise limit formula.

Example 5.6. As in example 3.10, consider the right Kan extension of F along $1_{\mathcal{C}}$. Since $F = \mathrm{Ran}_{1_{\mathcal{C}}} F$ with $1_F = \epsilon$, by Corollary 4.4, for each $s \in \mathbf{Set}$ the functor $\mathbf{Set}(s, F-)$ is the right Kan extension of $\mathbf{Set}(s, F-)$ with counit $\mathbf{Set}(s, 1_F) =$

$1_{\mathbf{Set}(s,F-)}$. This simply says that the extension F along $1_{\mathcal{C}}$ is preserved by all representable functors, making $F = \mathrm{Ran}_{1_{\mathcal{C}}}F$ a pointwise right Kan extension.

Since F is a pointwise right Kan extension, we can use the limit formula for $\mathrm{Ran}_K F$ of Theorem 3.1. We get

$$Fc = (\mathrm{Ran}_{1_{\mathcal{C}}}F)c = \mathrm{Lim}((c \downarrow 1_{\mathcal{C}}) \xrightarrow{Q^c} \mathcal{C} \xrightarrow{F} \mathbf{Set})$$

Additionally, since F is pointwise, every limit of FQ^c exists, thus the set of cones over FQ^c is representable; specifically, cones over FQ^c are represented by the limit of FQ^c at each c, so

$$\mathbf{Set}(s, Fc) = \mathbf{Set}(s, \mathrm{Lim}FQ^c) \cong [(c \downarrow 1_{\mathcal{C}}), \mathbf{Set}](\Delta_s, FQ^c)$$

However, applying lemma 5.4, this is isomorphic to $[\mathcal{C}, \mathbf{Set}](\mathcal{C}(c,-), \mathbf{Set}(s, F-))$. Simplifying, we get

$$\mathbf{Set}(s, Fc) \cong [\mathcal{C}, \mathbf{Set}](\mathcal{C}(c,-), \mathbf{Set}(s, F-)) \tag{5.7}$$

for all s in **Set**.

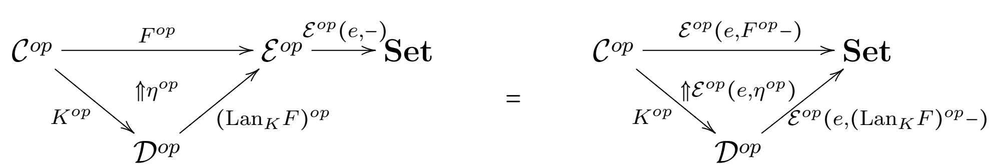
Since this isomorphism holds for any s, let $s = *$, the one element set. Since a map from the one element set to any other set s simply picks out one element of the set s, the set of maps $\mathbf{Set}(*, s) = s$. Using this property twice, and (5.7) we get

$$\begin{aligned} Fc &\cong \mathbf{Set}(*, Fc) \\ &\cong [\mathcal{C}, \mathbf{Set}](\mathcal{C}(c,-), \mathbf{Set}(*, F-)) \\ &\cong [\mathcal{C}, \mathbf{Set}](\mathcal{C}(c,-), F-) \end{aligned}$$

However, this is simply the Yoneda lemma!

Theorem 5.8. *A functor $F{:}\mathcal{C} \to \mathcal{E}$ has a pointwise left Kan extension if and only if $(K \downarrow d) \xrightarrow{\bar{Q}^d} \mathcal{C} \xrightarrow{F} \mathcal{E}$ has a colimit for each $d \in \mathcal{D}$. Then $\mathrm{Lan}_K F$ is given by the equation* (3.7)*.*

Proof. By definition, a left Kan extension of F along K is pointwise if and only if $\mathrm{Lan}_K F$ is preserved by $\mathcal{E}(-, e)$ for all $e \in \mathcal{E}$. By the duality of **Cat** and $\mathbf{Cat}^{co}$, this is occurs if and only if $(\mathrm{Lan}_K F)^{op}$ is a right pointwise Kan extension of F^{op} along K^{op}



However, by Theorem 5.5, $(\mathrm{Lan}_K F)^{op}$ is a pointwise right extension if and only if for each $d \in \mathcal{D}^{op}$,

$$(\mathrm{Lan}_K F)^{op} d = \mathrm{Lim}((d \downarrow K^{op}) \xrightarrow{Q^d} \mathcal{C}^{op} \xrightarrow{F^{op}} \mathcal{E}^{op})$$

If the limit exists for each d and is given by $(\mathrm{Lan}_K F)^{op}$, then the cones over $F^{op}Q^d$ are representable, so

$$[(d \downarrow K^{op}), \mathcal{E}^{op}](\Delta_e, F^{op}Q^d) \cong \mathcal{E}^{op}(e, (\mathrm{Lan}_K F)^{op} d)$$

Applying the functor $(-)^{op} \colon \mathbf{Cat} \to \mathbf{Cat}^{co}$ to both sides, again by the duality of $\mathbf{Cat}$ and $\mathbf{Cat}^{co}$, yields

$$[(K \downarrow d), \mathcal{E}](F\bar{Q}^d, \Delta_e) \cong \mathcal{E}(\mathrm{Lan}_K F d, e)$$

This isomorphism says that for each $d \in \mathcal{D}$, the $\mathrm{Lan}_K F d$ represents the cones under $F\bar{Q}^d$, that is, $\mathrm{Lan}_K F d = \mathrm{Colim}(F\bar{Q}^d)$, which is exactly the formula of equation (3.7). Thus a left Kan extension is pointwise if and only if the colimit of $F\bar{Q}^d$ exists for all $d \in \mathcal{D}$. □

6 Density

Again making use of the limit and colimit formulas, this section will relate Kan extensions to the notion of density. First, we define density and codensity through limits and colimits. Then we use the limit formula of Theorem 3.1 for pointwise Kan extensions to show that a functor being codense is equivalent to certain conditions on the right Kan extension of the functor along itself. Drawing out this notion, we will prove that the Yoneda embedding is dense. Finally, we will tease out what exactly it means for the Yoneda embedding to be dense, showing that every presheaf is a canonical colimit of representable functors over the category of elements of the presheaf.

Definition 6.1. A functor $K \colon \mathcal{C} \to \mathcal{D}$ is *dense* if for each $d \in \mathcal{D}$

$$\mathrm{Colim}((K \downarrow d) \xrightarrow{\bar{Q}^d} \mathcal{C} \xrightarrow{K} \mathcal{D}) = d$$

with canonical colimiting cone.

Since the elements of $(K \downarrow d)$ are pairs $(c \in \mathcal{C}, f \colon Kc \to d)$, each object of the comma category is already equipped with a map $Kc \to d$, as is needed in a colimiting cone, making the maps f the canonical cone.

Dually, a functor is *codense* when, for each $d \in \mathcal{D}$

$$\mathrm{Lim}((d \downarrow K) \xrightarrow{Q^d} \mathcal{C} \xrightarrow{K} \mathcal{D}) = d$$

again with the canonical limiting cone.

Theorem 6.2. *A functor $K{:}\mathcal{C} \to \mathcal{D}$ is condense if and only if $1_{\mathcal{D}}$ together with 1_K is the pointwise right Kan extension of K along K.*

Proof. If K is codense, then for each d, $\mathrm{Lim} KQ^d$ exists. By Theorem 5.5, K has a pointwise right Kan extension along K, given by the limit formula. However, $(\mathrm{Ran}_K K)d = \mathrm{Lim} KQ^d = d$, for all $d \in \mathcal{D}$ so the right Kan extension is given by $\mathrm{Ran}_K K = 1_{\mathcal{D}}$. Conversely, if K has $1_{\mathcal{D}}$ as a pointwise right Kan extension along K, then again by Theorem 5.5, $\mathrm{Lim} KQ^d = (\mathrm{Ran}_K K)d = d$, so K is condense, as desired. □

Dually, a functor is dense if and only if $(1_{\mathcal{D}}, 1_K)$ is the pointwise left Kan extension of K along K, since the colimits of Definiton 6.1 are exactly the colimits in the formula of (3.7) and Theorem 5.8.

Corollary 6.3. *A functor $K{:}\mathcal{C} \to \mathcal{D}$ is condense if and only if $f \mapsto D(f, K-)$ is for all $d, d' \in \mathcal{D}$ a bijection*

$$\mathcal{D}(d, d') \cong [\mathcal{C}, Set](\mathcal{D}(d', K-), \mathcal{D}(d, K-))$$

That is if and only if the functor $\mathcal{D}^{op} \to [\mathcal{C}, \mathbf{Set}]$ is fully faithful.

Proof. In the proof of Theorem 5.5, we showed a pointwise right Kan extension gives the isomorphism

$$\mathcal{E}(e, \mathrm{Ran}_K Fd) \cong [\mathcal{C}, \mathbf{Set}](\mathcal{D}(d', K-), \mathcal{E}(e, F-))$$

Theorem 6.2 allows us to replace F with K and $\mathrm{Ran}_K F$ with $1_{\mathcal{D}}$. Thus K is codense if and only if

$$\mathcal{D}(d, d') \cong [\mathcal{C}, \mathbf{Set}](\mathcal{D}(d', K-), \mathcal{D}(d, K-))$$

Consider the functor which maps $d \in \mathcal{D}$ to the functor $\mathcal{D}(d, K-) \in [\mathcal{C}, \mathbf{Set}]$. The isomorphism above gives that for each map $f{:}d \to d'$ in $\mathcal{D}$, there is a unique map $f^*{:}\mathcal{D}(d', K-) \to \mathcal{D}(d, K-)$ in $[\mathcal{C}, \mathbf{Set}]$, or that the functor $\mathcal{D}^{op} \to [\mathcal{C}, \mathbf{Set}]$ is fully faithful. □

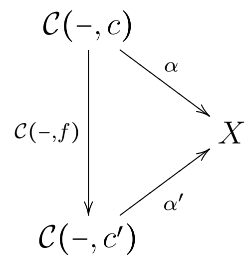
Corollary 6.4. *The Yoneda embedding $Y{:}\mathcal{C} \to [\mathcal{C}^{op}, \mathbf{Set}]$ is dense.*

Proof. By the Yoneda Lemma, the functor $Y^{op}: \mathcal{C}^{op} \to [\mathcal{C}^{op}, \mathbf{Set}]^{op}$ is fully faithful, and so is codense. Thus the right Kan extension of Y^{op} along itself is the identity functor $1_{[\mathcal{C}^{op},\mathbf{Set}]^{op}}$. Applying the functor $(-)^{op}: \mathbf{Cat} \to \mathbf{Cat}^{co}$ tells us that the identity functor $1_{[\mathcal{C}^{op},\mathbf{Set}]}$ is the left Kan extension of Y along itself, making Y dense. □

Often, this corollary is called the Density Theorem, or the co-Yoneda theorem. What this corollary says specifically is that

$$\mathrm{Colim}((Y \downarrow X) \xrightarrow{\bar{Q}^X} \mathcal{C}^{op} \xrightarrow{Y} [\mathcal{C}, \mathbf{Set}] = X$$

for $X \in [\mathcal{C}^{op}, \mathbf{Set}]$. Recall that objects in $(Y \downarrow X)$ are pairs $(c \in \mathcal{C}, \alpha: \mathcal{C}(-, c) \Rightarrow X)$ with maps $f: c \to c'$ such that the following triangle commutes:



By the Yoneda lemma, these pairs (c, α) are equivalent to pairs $(c, x \in X(c))$ with maps $(c, x) \to (c', x')$ being maps $f: c \to c'$ such that $(Xf)(x') = x$. These pairs are objects of the category of elements $E(X)$ of X, with the projection function $\bar{Q}^X$ being equivalent to the projection $P: E(X) \to \mathcal{C}$ which maps (c, x) to c and f to f. Thus, Y being dense is equivalent to saying that each presheaf X is a canonical colimit of the diagram $E(X) \xrightarrow{P} \mathcal{C} \xrightarrow{Y} \mathbf{Set}$, or the colimit of representable functors.

7 Formal Category Theory

The proofs given in this section can be interpreted in any 2-category with a terminal object. Rather than give new definitions, we continue to work in the 2-category **Cat**. Our aim is to give new proofs of "classical" categorical results using Kan extensions. We will show that adjoints preserve limits, and that adjoints compose, using the theorems and examples of the previous sections.

Theorem 7.1. *Left adjoints preserve colimits.*

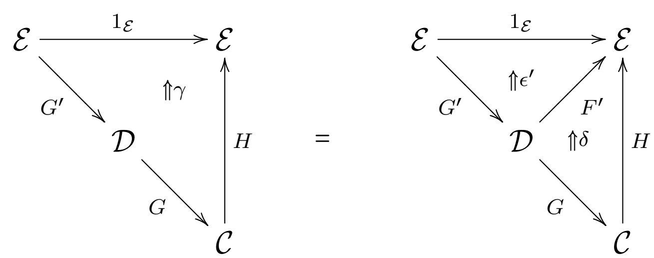
Proof. By Theorem 4.2, left adjoints preserve left Kan extensions. By example 2.4, colimits are just the left Kan extension of a functor through the terminal category. Thus, left adjoints preserve colimits. □

Dually, right adjoints preserve limits, since limits are right Kan extensions through the terminal category.

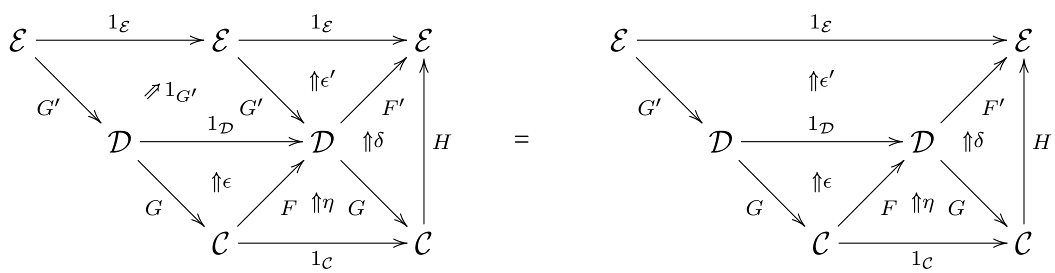
Theorem 7.2. *If* $F: \mathcal{C} \to \mathcal{D}, G: \mathcal{D} \to \mathcal{C}$ *with* $F \dashv G$ *and* $F': \mathcal{D} \to \mathcal{E}, G': \mathcal{E} \to \mathcal{D}$ *with* $F' \dashv G'$*, then* $F'F \dashv GG'$*. In other words, adjunctions compose.*

Proof. Assume $F \dashv G$ with unit η and counit ϵ, and $F' \dashv G'$ with unit η' and counit ϵ'. We will show $F'F \dashv GG'$ with counit $\epsilon' \circ F'\epsilon G'$ by appealing to Theorem 4.5 and proving that $F'F$ is a right Kan extension of $1_{\mathcal{E}}$ along GG', and that this extension is preserved by GG'. By Theorem 4.5 $F = \mathrm{Ran}_G 1_{\mathcal{D}}$, and is preserved by G, while $F' = \mathrm{Ran}_{G'} 1_{\mathcal{E}}$ and is preserved by G'.

Let $H: \mathcal{C} \to \mathcal{E}$ with $\gamma: HGG' \Rightarrow 1_{\mathcal{E}}$. Then, since $F' = \mathrm{Ran}_{G'} 1_{\mathcal{E}}$, there exists a unique $\delta: HG \Rightarrow F'$ which factors γ through ϵ' as shown.

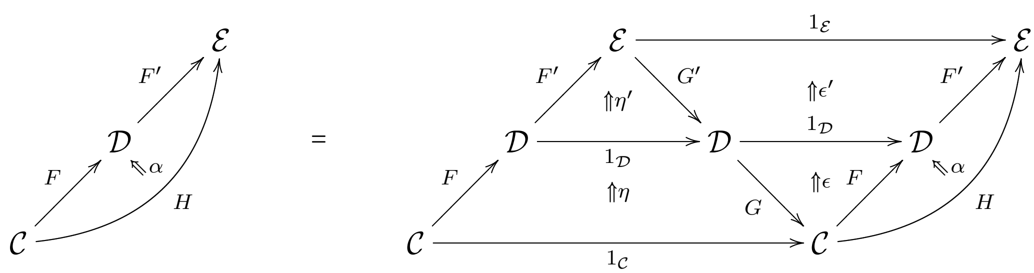


Then, by one of the triangle identities of $F \dashv G$, the right hand side is equivalent to

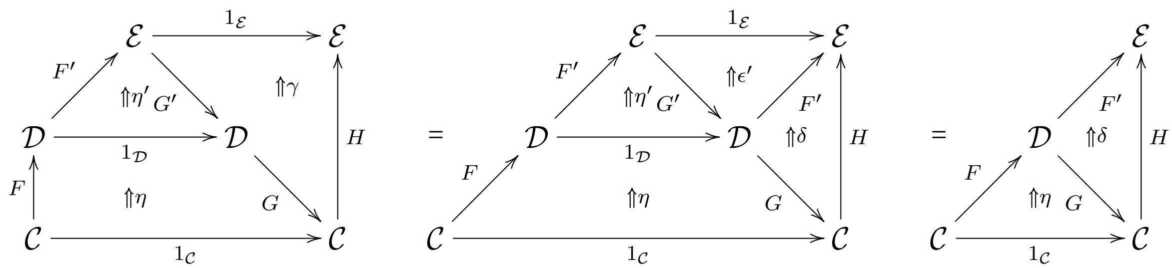


This gives a factorization of the pair (H, γ) through $F'F$.

To show this factorization is unique, consider $\alpha: H \Rightarrow F'F$, a factorization of (H, γ). Then by the triangle identities for $F \dashv G$ and $F' \dashv G'$,



Then, since α was a factorization of (H, γ), and since we can factor γ uniquely through δ, this is equivalent to



where the last equality holds thanks to a triangle identity from $F' \dashv G'$. Thus, the factorization of γ through $F'F$ is unique, so $F'F$ is the right Kan extension of $1_{\mathcal{E}}$ along GG'.

Since G' is a right adjoint, it will preserve the right Kan extension $F'F$. Similarly, since G is a right adjoint, it will preserve the right Kan extension $G'F'F$. Thus, the composite functor GG' preserves the right Kan extension $F'F$. By Theorem 4.5, this means that $F'F$ is the left adjoint of GG', so adjunctions compose. $\square$

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