# Classifying Varieties with Many Lines

A Senior Thesis by Silas Richelson

Advisor: Joe Harris

submitted in partial fulfillment of the honors requirements for the degree of bachelor of arts

Harvard University Math Department
Cambridge, MA
March 31, 2008

# Contents

## Acknowledgements

I would like to thank my advisor Professor Joe Harris. Throughout our time working together, he was incredibly helpful and encouraging. He provided an excellent environment for me to learn algebraic geometry at my own pace. In addition to meeting with me weekly for three semesters and a summer, he would frequently extend our meetings longer than the scheduled hour to fully answer whatever questions I had. He was always generous with his time, he was helpful, and he was never impatient. It was truly a pleasure for me to be able to work with him for so long, and I consider myself lucky to have been allowed to do so.

Additionally, I would like to thank my parents for their constant love and support.

# Preface

One of the amazing features of algebraic geometry is the number of possible pitfalls to fall into while learning the subject for the first time. If one attempts to approach the subject in full generality, it will likely be difficult to build any intuition from the overwhelming mass of technical machinery. If, on the other hand, one attempts to approach the subject more classically, by examining many only slightly related examples, it could be difficult to recognize the key results amid the barrage of isolated, witty arguments. The real problem, it seems, is that there is no clear way to develop the intuition necessary to study the subject at any sort of deep level. This dilemma can make algebraic geometry an especially difficult subject to learn. I was no exception. I frequently would spend hours attempting to read through proofs in which the author would implicitly assume facts which are obvious given the proper intuition, but which otherwise require careful verification. In order to avoid complaining about, and subsequently duplicating these practices, one of the main goals of this thesis is to make all proofs completely rigorous and to avoid the temptation to hand-wave and appeal to intuition whenever possible. I apologize in advance if this policy, at times, becomes overkill. I decided that it was better, especially for a senior thesis, to err on the side of explanation. Plus, this is likely to be one of the only times in my life when I won't have a referee requiring me to shorten my writing and so I plan to take full advantage.

# 1 Introduction

The problem of classifying algebraic varieties up to isomorphism is at the very heart of algebraic geometry. Many classical notions, such as irreducibility, the coordinate ring, dimension, smoothness, the Picard group, and genus (for curves) can be associated to a variety, and are invariant under isomorphism. Many statements about the classification of varieties can be proved using these, and other classical notions. Del Pezzo, Kronecker, Corrado Segre and other members of the Italian school proved many such results. More recently, the development of cohomology has allowed for deep classification theorems that had eluded proof using classical techniques.

Another invariant of a variety is its set of subvarieties. On its own, the set of subvarieties is generally too large and unmanageable to be used to effectively classify anything, however, we can frequently gain insight from certain families of subvarieties. The most basic example of this, and the one with which we will concern ourselves in this paper, is the problem of determining $X$ explicitly given the dimension of the set of lines contained in $X$, denoted $F_1(X)$. In particular, we will be studying $X$ for which $\dim F_1(X)$ is large, as it is a far easier problem than when $\dim F_1(X)$ is small. Indeed, the more lines a variety contains, the more of its structure is reflected in its Fano variety. The question of what is meant by large is easily answered. As we will soon see, if $\dim X = k$, then $\dim F_1(X) \leq 2k - 2$ with equality holding if and only if $X$ is a $k$-plane. One natural follow-up question is 'What if $\dim F_1(X) = 2k - 3$?' This is a classical result due to Beniamino Segre (see [S]). He proves that such an $X$ is either a one-parameter family of $\mathbb{P}^{k-1}$ or a quadric hypersurface. The next case, the one where $\dim F_1(X) = 2k - 4$, is the last case to be studied in any detail. In 1994, Enrico Rogora proved a partial classification theorem (see [R1]). He proved that if $X \subset \mathbb{P}^n$ with $\dim F_1(X) = 2k - 4$ has codimension greater than two, and is swept out by its Fano variety, then either $X$ is a two-parameter family of $\mathbb{P}^{k-2}$, a one-parameter family of quadrics, or it is a linear section of $\mathbb{G}(1, 4)$, the Grassmannian of lines in $\mathbb{P}^4$. In 2005, Landsberg and Robles handled the codimension 2 case under the Fubini hypothesis, which states that any line meeting $X$ at a general point with multiplicity at least two, meets with multiplicity at least three (see [L-R]).

# 2 Background

## 2.1 Author's Note

One of the funny things about classical algebraic geometry is that, more than most other subjects, the list of slightly nontrivial facts is overwhelmingly extensive. This presents one wishing to write a classical algebraic geometry paper with a dilemma. On the one hand, it would be nice to be able to thoroughly explain every detail. On the other hand, however, too much explanation would likely upset the paper's flow. In order to solve this problem, we list several of the classical results we will be assuming, sometimes implicitly, in the arguments to follow. We will leave out the proofs as the majority of them are either uninteresting or completely standard.

## 2.2 Miscellaneousness

**Fact.** If $X \subset \mathbb{P}^n$ is an irreducible variety of dimension at least 2, then for the general hyperplane $H \subset \mathbb{P}^n$, the intersection $H \cap X$ is irreducible.

**Fact.** If $X \subset \mathbb{P}^n$ is a nondegenerate variety then for a general hyperplane $H \subset \mathbb{P}^n$, $X \cap H$ is nondegenerate.

**Fact.** A cone with more than one vertex is a plane.

**Fact.** If $X$ is a cone with vertex $p$, then the tangent space to $X$ along a general line of $X$ through $p$ is fixed (ie: $\mathbb{T}_q X = \mathbb{T}_{q'} X$ for general $q, q' \in l$, where $l$ is a general line of $X$ through $p$).

**Fact.** If $X \subset \mathbb{P}^n$ is a variety and $Y \subset X$ a subvariety, then for every $p \in Y$, $\mathbb{T}_p Y \subset \mathbb{T}_p X$.

**Fact.** If $X \subset \mathbb{P}^n$ is a $k$-dimensional variety, then for the general choice of hyperplanes $H_1, \ldots, H_{k-r} \subset \mathbb{P}^n$, the variety

$$X' = X \cap H_1 \cap \cdots \cap H_{k-r}$$

will have dimension $r$.

**Fact.** If $Z \subset \mathbb{P}^n$ is a variety and $X, Y \subset Z$ are intersecting subvarieties, then

$$\dim(X \cap Y) \geq \dim X + \dim Y - \dim Z.$$

**Fact.** Let $X \subset \mathbb{P}^n$ be a variety and let $p \in \mathbb{P}^n$ be any point. Let

$$\pi_p : X \dashrightarrow \mathbb{P}^{n-1}$$

be projection through $p$ (note that $\pi_p$ is regular when $p \notin X$). Let $Y = \pi_p(X) \subset \mathbb{P}^{n-1}$. Then

1. if $p \notin X$ then $\deg Y = \deg X$;

2. if $p \in X$ is a smooth point then $\deg Y = \deg X - 1$;
3. if $p \in X$ is a singular point of multiplicity $m$ then $\deg Y = \deg X - m$.

**Fact.** If $p \in X \subset \mathbb{P}^n$ is contained in $N$ distinct irreducible components of $X$ counting multiplicity and $H \subset \mathbb{P}^n$ is a hyperplane containing $p$, then $X$ and $H$ intersect at $p$ with multiplicity at least $N$.

**Fact.** Let $\varphi : X \to Y \subset \mathbb{P}^m$ be a surjective, finite regular map. Let $\deg_0(\varphi)$ be the projective degree of $\varphi$ (ie: the number of points in the preimage of a general $(m - \dim X)$-plane in $\mathbb{P}^m$), and let $\deg(\varphi)$ be the degree of $\varphi$ (ie: the number of points in the preimage of a general $y \in Y$). Then

$$\deg_0(\varphi) = \big(\deg Y\big)\big(\deg(\varphi)\big).$$

**Fact.** If $X \subset \mathbb{P}^n$ is a nondegenerate variety then $\deg X \geq n - \dim X + 1$.

## 2.3 Some Bigger Results

We will use the following standard result, akin to the rank-nullity theorem from linear algebra, constantly.

**Theorem** (Theorem of the Fibers)**.** Let $X \subset \mathbb{P}^n$ be an irreducible variety and let $\varphi : X \to \mathbb{P}^m$ be a regular map with $Y = \varphi(X) \subset \mathbb{P}^m$. Then for every $y \in Y$, we have

$$\dim X \leq \dim Y + \dim \varphi^{-1}(y),$$

with equality holding for general $y \in Y$.

We will also find occasion to use Bézout's theorem.

**Theorem** (Bézout's Theorem)**.** If $X, Y \subset \mathbb{P}^n$ are intersecting varieties and $Z_1, \ldots, Z_N$ are the components of the intersection, each $Z_i$ occuring with multiplicity $m_i$, then

$$(\deg X)(\deg Y) = \sum_{i=1}^{N} m_i \deg Z_i.$$

Both of these results are proved in [Shaf].

# 3 The Grassmannian

For lack of a more natural starting point, we begin by examining the Grassmannian.

## 3.1 The Grassmannian as a Variety

Let $V$ be a vector space. The Grassmannian $\mathrm{G}r(k, V)$ is defined as the set of $k$-dimensional subspaces of $V$. When $V = \mathbb{A}^n$, as it always will for our purposes, we write $\mathrm{G}r(k, n)$ instead of $\mathrm{G}r(k, \mathbb{A}^n)$. We denote by $\mathbb{G}(k, n)$ the set of $k$-planes in $\mathbb{P}^n$. Since a $k$-plane through the origin in $\mathbb{A}^n$ is the same as a $(k-1)$-plane in $\mathbb{P}^{n-1}$, we have a natural identification of $\mathbb{G}(k-1, n-1)$ with $\mathrm{G}r(k, n)$.

In order to give $\mathrm{G}r(k, n)$ the structure of a projective variety we must first describe a way to embed it in a projective space. We do so using the plücker embedding.

**Definition.** Define the map

$$\Phi : \mathrm{G}r(k, n) \to \mathbb{P}\big(\wedge^k V\big) = \mathbb{P}^N : R \mapsto \lambda = v_1 \wedge \cdots \wedge v_k,$$

where $N = \binom{n}{k} - 1$ and $B = \{v_1, \dots, v_k\}$ is a basis for $R$. The map $\Phi$ is called the *plücker embedding.*

**Note.** The plücker embedding is well defined since if $\{v_1', \dots, v_k'\}$ is another basis for $R$ then

$$v_1' \wedge \cdots \wedge v_k' = \alpha\big(v_1 \wedge \cdots \wedge v_k\big)$$

where $\alpha$ is the determinant of the change of basis matrix $A : v_i' \mapsto v_i$. Therefore, choosing a different basis for $R$ multiplies the corresponding $k$-form by a nonzero element of the ground field $K$, and so the point in projective space remains unchanged.

**Note.** The plücker embedding is injective since given any $\lambda \in \mathrm{Im}\Phi$ we may recover its preimage as the kernal of the linear map

$$\varphi : \mathbb{A}^n \to \wedge^{k+1}\mathbb{A}^n : v \mapsto v \wedge \lambda.$$

That $\Phi$ is injective allows us to identify $\mathrm{G}r(k, n)$ with a subset of $\mathbb{P}^N$ which is, in fact, a subvariety due to the following claim.

**Claim.** $\mathrm{G}r(k, n) \simeq \mathrm{Im}(\Phi) \subset \mathbb{P}^N$ is closed.

*Proof.* Given any $[\lambda] \in \mathbb{P}^N$, define the linear map

$$\varphi_\lambda : \mathbb{A}^n \to \wedge^{k+1}\mathbb{A}^n : v \mapsto v \wedge \lambda.$$

Notice that $\dim\big(\ker\varphi_\lambda\big) = k$ if and only if $\lambda$ can be written as the wedge product of $k$-linearly independent vectors, $\dim\big(\ker\varphi_\lambda\big) < k$ otherwise. Additionally, note that $[\lambda] \in \mathrm{Im}\Phi$ if and only if $\lambda$ can be written as the wedge product of $k$

linearly independent vectors. Therefore, we see that $[\lambda] \in \mathrm{Im}\Phi$ if and only if $\mathrm{rank}(\varphi_\lambda) = n-k \Leftrightarrow \mathrm{rank}(\varphi_\lambda) \leq n-k$ (since as noted above, if $\mathrm{rank}(\varphi_\lambda) \neq n-k$ then $\mathrm{rank}(\varphi_\lambda) > n-k$). However, this tells us that $\mathrm{Im}\Phi$ is closed since $[\lambda] \in \mathrm{Im}\Phi$ if and only if the $(n-k) \times (n-k)$ minors of the matrix of $\varphi_\lambda$ vanish. $\square$

Now that we have identified $\mathrm{G}r(k,n)$ with a projective subvariety via the plücker embedding $\Phi$, we immediately become lazy and speak of $\mathrm{G}r(k,n)$, itself as being a projective subvariety of $\mathbb{P}^N$, rather than $\Phi\big(\mathrm{G}r(k,n)\big)$. Similarly, we frequeltly speak of the $k$-plane $R \in \mathrm{G}r(k,n)$ as being a point in $\mathbb{P}^N$ rather than the unique preimage of the point $\Phi(R) \in \mathbb{P}^N$.

## 3.2 Dimension of the Grassmannian

We will determine the dimension of the Grassmannian by examining subsets

$$U_S = \big\{R \in \mathrm{G}r(k,n) : R \cap S = \{0\}\big\},$$

where $S \subset \mathbb{P}^n$ is any $(n-k)$-plane.

**Note.** Clearly $U_S \subset \mathrm{G}r(k,n)$ is open since for an $k$-plane, $R$, to intersect $S$ nontrivially, it must be that the union of their bases is a linearly dependent set. This says exactly that the matrix formed by putting the vectors in this union as the columns will have determinant zero. Note that this matrix will be well defined up to conjugation by certain (not all) change of basis matrices, and so whether the determinant equals zero or not does not depend on the bases chosen.

Next, we show that the open set $U_S$ can be viewed (under a proper choice of basis) as an affine chart of $\mathrm{G}r(k,n)$.

**Claim.** For any $(n-k)$-plane $S$, we may choose a basis for $\wedge^k \mathbb{A}^n$ such that $U_S = \mathbb{A}_0^N \cap \mathrm{G}r(k,n)$.

*Proof.* First, we must decide on a choice of basis. This is easy enough, let $S \subset \mathbb{P}^n$ be an $(n-k)$-plane with basis $\mathcal{B} = \{v_{k+1}, \ldots, v_n\}$ and extend $\mathcal{B}$ to a basis $\mathcal{B}' = \{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ for $\mathbb{A}^n$. Next we reduce the problem to easy linear algebra. Pick any $k$-plane $R \in \mathrm{G}r(k,n)$ spanned by $\{w_1, \ldots, w_k\}$ with

$$w_j = \sum_i \alpha_{ij} v_i.$$

If we express the $k$-form $w_1 \wedge \cdots \wedge w_k$ in terms of the basis $\{v_{m_1} \wedge \cdots \wedge v_{m_k}\}$ for $\wedge^k \mathbb{A}^n$ we have

$$w_1 \wedge \cdots \wedge w_k = \sum_{1 \leq m_1 < \cdots < m_k \leq n} \beta_{m_1,\ldots,m_k} v_{m_1} \wedge \cdots \wedge v_{m_k},$$

where $\beta_{m_1,\ldots,m_k} = \det(\alpha_{m_i m_j})$. Therefore, we see that $R \notin \mathbb{A}_0^N$ if and only if

$$\det \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1k} \\ \vdots & \ddots & \vdots \\ \alpha_{k1} & \cdots & \alpha_{kk} \end{pmatrix} = 0,$$

which happens if and only if the vectors

$$u_i = \begin{pmatrix} \alpha_{1i} \\ \vdots \\ \alpha_{ki} \end{pmatrix}$$

are linearly dependent. Therefore, it suffices to show that the $k$-plane, $R$, spanned by $\{w_1, \ldots, w_k\}$, with $w_j = \sum_i \alpha_{ij} v_i$, intersects $S$ nontrivially if and only if the vectors $u_1, \ldots, u_k$ are linearly dependent. This is clear. The $k$-plane $R$ will intersect $S$ nontrivially, if and only if there is a linear combination of the $w_j$ such that when the linear combination is written in terms of the $v_i$, the coefficients of $v_1, \ldots, v_k$ are zero (since only then will the vector lie in $S = \mathrm{Span}\{v_{k+1}, \ldots, v_n\}$). By construction of the $u_i$, this happens exactly when $\{u_i, \ldots, u_k\}$ is a linearly dependent set. □

Since the affine charts of the Grassmannian have the form $U_S$ for some $(n-k)$-plane $S$, much of the structure of the Grassmannian can be ascertained from the structure of $U_S$. We, therefore, study $U_S$ in greater detail. In particular, we prove the following.

**Claim.** For any $R_0 \in U_S$, we have

$$U_S \simeq \mathrm{Hom}(R_0, S) \simeq \mathbb{A}^{k(n-k)}$$

(the isomorphisms being isomorphisms of affine varieties).

*Proof.* For any $A \in \mathrm{Hom}(R_0, S)$ define the linear map $\varphi_A$ by $\varphi_A(v) = Av + v$. Note that $\varphi_A$ is injective since if $\varphi_A(v) = \varphi_A(v')$ then we have $Av - Av' = v - v'$, which means a vector in $S$ must equal a vector in $R_0$. Since $S \cap R_0 = \{0\}$ we must have $v = v'$. Therefore, $\mathrm{Im}(\varphi_A)$ is a $k$-plane which intersects $S$ only at $\{0\}$. Therefore we have a map

$$\begin{aligned} \psi : \quad & \mathrm{Hom}(R_0, S) \to U_S \\ & A \mapsto \mathrm{Im}(\varphi_A). \end{aligned}$$

Clearly $\psi$ is injective since if $\mathrm{Im}(\varphi_A) = \mathrm{Im}(\varphi_B)$ then for every $v \in R_0$ there exists a $v' \in R_0$ such that $\varphi_A(v) = \varphi_B(v')$ which gives us $Av - Bv' = v - v'$, and so again we have $v = v'$ and so $Av = Bv$ for every $v$.

We now show that $\psi$ is surjective. Since $R_0$ and $S$ intersect trivially and are of complimentary dimension, $\{v_1, \ldots, v_n\}$ is a basis for $\mathbb{A}^n$ where $\{v_1, \ldots, v_k\}$ is a basis for $R_0$ and $\{v_{k+1}, \ldots, v_n\}$ is a basis for $S$. Therefore, for any $R \in U_S$, any $v \in R$ can be written uniquely as

$$v = (\alpha_1 v_1 + \cdots + \alpha_k v_k) + (\alpha_{k+1} v_{k+1} + \cdots + \alpha_n v_n) = v_{R_0} + v_S,$$

and so we may define linear maps

$$\pi_{R_0} : R \to R_0 : v \mapsto v_{R_0}; \text{ and } \pi_S : R \to S : v \mapsto v_S.$$

Note that $\pi_{R_0}$ is injective since if there exist $v, v' \in R$ such that $v_{R_0} = v'_{R_0}$ then $v - v' = v_S - v'_S$. However, $v - v' \in R$ and $v_S - v'_S \in S$, and so $v = v'$ (since $R \cap S = \{0\}$). This allows us to define a linear map $A : R_0 \to S$ such that the following diagram commutes

$$\begin{tikzcd} & R \arrow[dl, "\pi_{R_0}"'] \arrow[dr, "\pi_S"] & \\ R_0 \arrow[rr, "A"] & & S \end{tikzcd}$$

This $A$ is an element of $\operatorname{Hom}(R_0, S)$ with the property that $\operatorname{Im}(\varphi_A) = R$. Therefore, $\psi(A) = R$, and $\psi$ is surjective.

Finally, note that $\psi$ is a regular map with a regular inverse. It is clearly regular since if we choose a basis a basis $\{w_1, \ldots, w_k\}$ for $R_0$ then

$$\psi(A) = Aw_1 \wedge \cdots \wedge Aw_k.$$

The inverse map is also regular since given $R$ which intersects $S$ trivially, we simply define $\psi^{-1}(R) = \pi_S \circ \pi_{R_0}^{-1}$. □

**Note.** This tells us that $U_S$ is irreducible, and has dimension $k(n-k)$.

One easy lemma, and then we are ready to compute the dimension, and prove the irreducibility of $\mathrm{G}r(k, n)$.

**Lemma.** Let $X \subset \mathbb{P}^n$ be a projective variety, and let $U_1, \ldots, U_N \subset X$ be open, irreducible and satisfying $U_i \cap U_j \neq \emptyset$ for all $i, j$. Then $X$ is irreducible.

*Proof.* Suppose $X = X_1 \cup X_2$. Then for each $i$, $U_i = (U_i \cap X_1) \cup (U_i \cap X_2)$. Therefore, either $U_i = U_i \cap X_1$ or else $U_i = U_i \cap X_2$. If $U_i \neq U_i \cap X_2$ for some $i$ then clearly $U_i = U_i \cap X_1$. Therefore, $X_1$ contains $U_i \cap U_j$ for all $j$. As each $U_i \cap U_j$ is nonempty, $X_1$ contains an open dense subset of $U_j$ for each $j$. Since $X_1$ is closed, it must be then that $X_1$ contains $U_j$ for all $j$. Therefore, since $X = U_1 \cup \cdots \cup U_N$, $X_1 = X$, and so $X$ is irreducible, as desired. □

**Proposition.** The Grassmannian $\mathrm{G}r(k, n)$ is irreducible of dimension $k(n-k)$.

*Proof.* As we have already seen, the Grassmannian is covered by open sets of the form $U_S$, each of which is irreducible. Therefore, in order to use the lemma we simply must remark that for any two $(n-k)$-planes, $S, S' \subset \mathbb{A}^n$, there exists a $k$-plane $\Gamma \subset \mathbb{A}^n$ such that $\Gamma \cap \big(S \cup S'\big) = \{0\}$. This last fact can be proven, for example, by an easy induction on $k$. □

**Note.** The Grassmannian of $k$-planes in $\mathbb{P}^n$ is an irreducible variety of dimension $(k+1)(n-k)$, since we have a natural identification of $\mathbb{G}(k, n)$ with $\mathrm{G}r(k+1, n+1)$.

## 3.3 A Useful Calculation

At this point, we halt to make a calculation that we use will use implicitly throughout this paper. For any $r$-plane $\Gamma_0 \subset \mathbb{P}^n$, $r \leq k$, we calculate $\dim \mathbb{G}_{\Gamma_0}$ where

$$\mathbb{G}_{\Gamma_0} = \{\Lambda \in \mathbb{G}(k,n) : \Gamma_0 \subset \Lambda\}.$$

To do this, we consider the variety

$$Z = \{(\Gamma, \Lambda) : \Gamma \subset \Lambda\} \subset \mathbb{G}(r,n) \times \mathbb{G}(k,n).$$

This is a variety because if $\{v_1, \ldots, v_r\}$ is a basis for $\Gamma$, and $\lambda$ is a $k$-form corresponding to $\Lambda$, then a the pair $(\Gamma, \Lambda)$ is in $Z$ if and only if $v_i \wedge \lambda = 0$ for all $i = 1, \ldots, r$, which gives equations in the plücker coordinates of $\mathbb{G}(r,n)$ and $\mathbb{G}(k,n)$. Let $\pi_1 : Z \to \mathbb{G}(r,n)$ and $\pi_2 : Z \to \mathbb{G}(k,n)$ be the projection maps. Clearly the $\pi_i$ are surjective. Therefore, by the theorem of the fibers we have that for general $\Gamma_0 \in \mathbb{G}(r,n)$ and $\Lambda_0 \in \mathbb{G}(k,n)$,

$$\dim \mathbb{G}(r,n) + \dim \mathbb{G}_{\Gamma_0} = \dim Z = \dim \mathbb{G}(k,n) + \dim \pi_2^{-1}(\Lambda_0).$$

Note that for any $\Lambda_0 \in \mathbb{G}(k,n)$,

$$\pi_2^{-1}(\Lambda_0) \simeq \{\Gamma \in \mathbb{G}(r,n) : \Gamma \subset \Lambda_0\} \simeq \mathbb{G}(r,k).$$

Therefore, for general $\Gamma_0 \in \mathbb{G}(r,n)$, we have

$$\dim \mathbb{G}_{\Gamma_0} = \dim \mathbb{G}(k,n) + \dim \mathbb{G}(r,k) - \dim \mathbb{G}(r,n) = (k-r)(n-k).$$

Finally, note that this holds for all $\Gamma_0 \in \mathbb{G}(r,n)$ since for any $\Gamma_0, \Gamma_1 \in \mathbb{G}(r,n)$ any of the projective lienar maps mapping $\Gamma_0$ to $\Gamma_1$ will induce an isomorphism $\mathbb{G}_{\Gamma_0} \simeq \mathbb{G}_{\Gamma_1}$.

## 3.4 The Fano Variety

For any set $S \subset \mathbb{P}^n$, we define

$$F_r(S) = \{\Lambda \in \mathbb{G}(r,n) : \Lambda \subset S\}$$

to be the set of $r$-planes contained in $S$.

**Note.** Clearly $F_r(S \cap S') = F_r(S) \cap F_r(S')$.

**Claim.** If $X \subset \mathbb{P}^n$ is a projective variety then $F_r(X)$ is a subvariety of the Grassmannian $\mathbb{G}(r,n)$.

*Proof.* By the above note, it suffices to prove the claim for $X \subset \mathbb{P}^n$ a hypersurface. Let $\mathbb{P}^N$ be the projective space parametrizing all degree $d$ hypersurfaces in $\mathbb{P}^n$ (ie: the projectivization of the vector space whose basis is the set of degree $d$ monomials in the variables $S_0, \ldots, S_n$). Then any hyperplane, $H$, in $\mathbb{P}^N$

corresponds to a hypersurface $\phi(H) \subset \mathbb{P}^n$. Given a hyperplane $H \subset \mathbb{P}^N$, let $\varphi_H$ be the polynomial that defines $\phi(H)$. Then define the set

$$Z = \{(\Lambda, H) : \Lambda \subset \phi(H)\} \subset \mathbb{G}(r, n) \times \mathbb{P}^N.$$

Once we show that $Z$ is a variety, it is clear that $F_r(X)$ is a variety since $F_r(\phi(H)) \simeq \pi_2^{-1}(H)$ where $\pi_2 : Z \to \mathbb{P}^N$ is projection onto the second component.

To see that $Z$ is a variety, note that $(\Lambda, H) \in Z$ if and only if $\varphi_H$ vanishes on all of $\Lambda$. We show that the vanishing of $\varphi_H$ on $\Lambda$ follows from the vanishing of $\varphi_H$ on finitely many points of $\Lambda$. This is a result of Bézout's theorem. Note that for any line in $\Lambda$, if $\varphi_H$ vanishes on $d+1$ points on the line, then it vanishes on the entire line (since $\deg \varphi_H = d$). Similarly, for any plane in $\Lambda$, if $\varphi_H$ vanishes on $d+1$ lines in the plane, then it vanishes on the entire plane (since for a general line in the plane, we would have $d+1$ vanishing points on the line, and so $\varphi_H$ must vanish on the general line of the plane). Therefore, in order to show that $\varphi_H$ vanishes on a plane of $\Lambda$ it suffices to check that $\varphi_H$ vanishes on $(d+1)^2$ points. Similarly (we may prove it easily by induction if we like), the vanishing of $\varphi_H$ on all of $\Lambda$ follows from the vanishing on $(d+1)^r$ points ($r = \dim \Lambda$). Finally, since the only requirement of these points is that they must be grouped into sets of $d+1$ which lie on the same line, and the $(d+1)^{r-1}$ resulting lines must be grouped into sets of $d+1$ which lie in a plane, and so on, we may choose $(d+1)^r$ such points in terms of any basis for $\Lambda$. Therefore, we have regular functions $x_i(\Lambda)$ for $i = 1, \ldots, (d+1)^r$ which denote the $(d+1)^r$ points of $\Lambda$. So finally, we see that $Z$ is a variety since it is cut out be the equations

$$\varphi_H\big(x_i(\Lambda)\big) = 0 : i = 1, \ldots, (d+1)^r,$$

and we are done. □

# 4 Our First Classification Result

## 4.1 A Maximum for $\dim F_r(X)$

**Claim.** Let $X \subset \mathbb{P}^n$ be a projective variety of dimension $k$, and let $H \subset \mathbb{P}^n$ be a hyperplane. Then as long as $H$ contains some $r$-plane that is contained in $X$,

$$\dim F_r(X \cap H) \geq \dim F_r(X) - r - 1.$$

*Proof.* Let $H \subset \mathbb{P}^n$ be a hyperplane containing some $r$-plane in $X$. Note that $F_r(H) = \mathbb{G}(r, n-1)$ and so $F_r(H)$ has codimension $r+1$ in $\mathbb{G}(r, n)$. Additionally, since $H$ contains an element of $F_r(X)$, the Fano variety of the intersection is nonempty. Therefore, we have

$$\begin{aligned}
\dim F_r(X \cap H) &= \dim\bigl(F_r(X) \cap F_r(H)\bigr) \\
&\geq \dim F_r(X) + \dim F_r(H) - \dim \mathbb{G}(k, n) \\
&= \dim F_r(X) - r - 1,
\end{aligned}$$

as desired. □

**Claim.** Let $X \subset \mathbb{P}^n$ be a projective variety of dimension $k$. Then

$$\dim F_r(X) \leq (r+1)(k-r).$$

*Proof.* We prove this by induction on $k$. When $k = r$, the claim is trivial since $X$ may contain at most finitely many $r$-planes, as each one would be an irreducible component. Therefore, $\dim F_r(X) \leq 0$ (we use the convention that $\dim \emptyset = -1$). Now suppose $\dim X = k > r$ and that $X$ contains at least one $r$-plane (since otherwise the result is clear). By the previous result, it suffices to find a hyperplane $H \subset \mathbb{P}^n$ such that $H$ contains an $r$-plane of $X$ and does not contain an irreducible component of $X$ of maximal dimension (since then $\dim(X \cap H) = k - 1$, and so the result follows by induction).

Clearly if no irreducible component of $X$ with maximal dimension is degenerate then any hyperplane will have the property that it does not contain a component of maximal dimension. The situation is only slightly more complicated when $X$ has degenerate components. Let $X_1 \subset X$ be an irreducible component of dimension $k$, and suppose that $X_1$ is degenerate, spanning an $m$-plane $\Gamma$. Furthermore, suppose $m$ is minimal. Notice that any hyperplane, $H$, not containing $\Gamma$ will not contain $X_1$, since otherwise $X_1$ would lie in the $(m-1)$-plane $\Gamma \cap H$ which contradicts the minimality of $m$. Finally, since $\dim X_1 = k > r$ we must have $m > r$. Therefore, finding a hyperplane with the desired property is equivalent to finding one which contains a given $r$-plane, but does not contain any element of a finite set of planes, each with dimension strictly greater than $r$. The latter is clearly possible, and so we are done. □

**Note.** This maximum is obtained. If $X \subset \mathbb{P}^n$ is a $k$-plane, for example, then $F_r(X) = \mathbb{G}(r, k)$ and so $\dim F_r(X) = (r+1)(k-r)$.

## 4.2 The Classification Theorem

In this section we prove the following theorem.

**Theorem 1.** If $X \subset \mathbb{P}^n$ is a projective variety of dimension $k$ such that $\dim F_r(X) = (r+1)(k-r)$ then $X$ contains a $k$-plane.

**Note.** We speak of $X$ *containing* a $k$-plane, rather than $X$ *being* a $k$-plane to allow for the possibility that $X$ is reducible. However, it is certainly true that any $k$-plane of $X$ must be an irreducible component. Furthermore, by the following observation, it suffices to consider the case when $X$ is irreducible.

**Note.** We clearly have

$$F_r(X_1 \cup X_2) = F_r(X_1) \cup F_r(X_2)$$

since for an $r$-plane, $\Lambda$, to be in $X_1 \cup X_2$ it must be that either $\Lambda \subset X_1$ or else $\Lambda \subset X_2$ since planes are irreducible. Therefore, if $X = X_1 \cup \cdots \cup X_m$ is a $k$-dimensional variety with $\dim F_r(X) = (r+1)(k-r)$ then it must be that $\dim F_r(X_i) = (r+1)(k-r)$ for some $i$. Furthermore, it must be that $\dim X_i = k$ since if $\dim X_i$ were any less, $X_i$ would violate the upper bound of the Fano variety found in the previous section. Therefore, theorem 1 follows from the following.

**Theorem.** If $X \subset \mathbb{P}^n$ is an irreducible, projective variety of dimension $k$ such that $\dim F_r(X) = (r+1)(k-r)$, then $X$ is a $k$-plane.

*Proof.* We prove this by induction on $r$. Because the proof is slightly long we break it up into two steps: the base case ($r = 1$) and the $r > 1$ case.

**Step 1:** $r = 1$

Let $X \subset \mathbb{P}^n$ be an irreducible $k$-dimensional variety such that $\dim F_1(X) = 2k-2$. Clearly $X$ is swept out by its Fano variety, since if the lines of $F_1(X)$ were to sweep out a proper subvariety, $X' \subset X$, it would be that $\dim X' \leq k-1$ but $\dim F_1(X') = 2k-2$ which violates the upper bound of $\dim F_r(X')$ obtained in the previous section. By using the theorem of the fibers on the variety

$$Z = \{(p, l) : p \in l\} \subset X \times F_1(X),$$

and its projections $\pi_1 : Z \to X$ and $\pi_2 : Z \to F_1(X)$, we get that for general $p \in X$, $\dim \Sigma_p = k-1$ where $\Sigma_p \subset F_1(X)$ is the family of lines through $p$. Let $X_p \subset X$ be the variety swept out by $\Sigma_p$. Clearly $X_p$ is a cone with vertex $p$, as it consists of lines through $p$. Now, consider the variety

$$Z_p = \{(q, l) : q \in l\} \subset X \times \Sigma_p,$$

and its projections $\pi_1$ and $\pi_2$. Clearly $\pi_1(Z_p) = X_p$. Also, for general $q \in X_p$, $\dim \pi_1^{-1}(q) = 0$, since $\pi_1^{-1}(q)$ is the set of lines in $\Sigma_p$ through $p$ and $q$ (and there is exactly one such line). Therefore, the theorem of the fibers gives

$$\dim X_p = \dim \Sigma_p + 1 = k,$$

and so since $X$ is irreducible, we see that $X_p = X$. But since $p \in X$ was an arbirary general point, we have that for general $p, q \in X$, $X = X_p = X_q$. This means that $X$ is a cone with more than one vertex. Thereore, $X$ is a plane.

**Step 2:** $r > 1$

When $r > 1$, given any hyperplane $H \subset \mathbb{P}^n$, we can define a map

$$\begin{aligned} \varphi_H : \quad & F_r(X) \dashrightarrow F_{r-1}(X \cap H) \\ & \Lambda \mapsto \Lambda \cap H. \end{aligned}$$

Clearly $\varphi_H$ is regular on the set of $r$-planes of $X$ which are not contained in $H$. Also note that

$$\varphi_H^{-1}(\Gamma) = \{\Lambda \in F_r(X) : \Gamma \subset \Lambda \not\subset H\}.$$

Let $\dim F_r(X) = (r+1)(k-r)$. If we can show that for general $H$, and general $\Lambda \in \mathrm{Im}\varphi_H$, $\dim \varphi_H^{-1}(\Gamma) \leq k - r$ we will be done because then we would have

$$\dim F_{r-1}(X \cap H) \geq \dim F_r(X) - k + r = r(k-r).$$

By induction, we have that $X \cap H$ is a $(k-1)$-plane (we know $X \cap H$ is irreducible since $H$ is general and $X$ is irreducible), and so the intersection of $X$ with a general hyperplane is a $(k-1)$-plane. Therefore, $\deg X = 1$ and so $X$ is a $k$-plane.

Let $H \subset \mathbb{P}^n$ be a general hyperplane, and let $\Gamma$ be a general $(r-1)$-plane in $\mathrm{Im}\varphi_H$. Since any $\Lambda \in \varphi_H^{-1}(\Gamma)$ must lie in $X$, for a general point $p \in \Lambda$, $\Lambda \subset \mathbb{T}_p X$. Since $X$ is swept out by the $r$-planes in its Fano variety (by reasoning as in the base case), and $\Gamma$ and $H$ were general, $p$ is a general point of $X$. Therefore, $\dim \mathbb{T}_p X = k$. This means that $\varphi_H^{-1}(\Gamma) \subset \mathbb{G}_\Gamma$ where $\mathbb{G}_\Gamma$ is the set of $r$-planes in $\mathbb{T}_p X$ which contain the $(r-1)$-plane $\Gamma$. By our useful calculation, $\dim \mathbb{G}_\Gamma = k-r$, and so $\dim \varphi_H^{-1}(\Gamma) \leq k - r$, as desired. $\square$

## 4.3 Discussion

This theorem is not surprising in the least. Clearly we would expect that of all $k$-dimensional varieties, a $k$-plane contains the most $r$-planes. In the upcoming section, we will attempt to classify the $k$-dimensional varieties whose Fano varieties have dimension near the maximum. It will not be surprising that such a variety is not far from a $k$-plane. Before we can prove the next classification theorem, however, we must develop the second fundamental form.

# 5 The Second Fundamental Form

## 5.1 Tangent Space of the Grassmannian

First, note that $\mathbb{G}(r,n)$ is smooth since for any two $r$-planes $\Lambda, \Lambda' \in \mathbb{G}(r,n)$, there is an automorphism of $\mathbb{G}(r,n)$ mapping $\Lambda$ to $\Lambda'$ (projective linear automorphisms of $\mathbb{P}^n$ induce automorphisms of $\mathbb{G}(r,n)$). Therefore, since automorphisms map singular points to singular points, if $\Lambda$ is a singular point of $\mathbb{G}(r,n)$, then every point of $\mathbb{G}(r,n)$ would be singular, which can't be the case.

Therefore, since $\dim \mathbb{G}(r,n) = (r+1)(n-r)$, for any $\Lambda \in \mathbb{G}(r,n) = \mathbb{G}$, $T_\Lambda \mathbb{G}$ will be a vector space of dimension $(r+1)(n-r)$.

If we want to be more explicit, recall that for any $(n-r)$-plane $S$ such that the affine chart, $U_S$, contains $\Lambda$, $U_S \simeq \mathrm{Hom}(\widetilde{\Lambda}, \widetilde{S})$, where $\widetilde{\Lambda} \subset \mathbb{A}^{n+1}$ is the $(r+1)$-plane lying above $\Lambda$ and $\widetilde{S}$ is the quotient of the $(n-r+1)$-plane lying above $S$ by its intersection with $\widetilde{\Lambda}$. Note that $\mathrm{Hom}(\widetilde{\Lambda}, \widetilde{S})$ is a vector space of dimension $(r+1)(n-r)$. Additionally, note that this construction needn't depend on the choice of $S$ since for every $(n-r)$-plane, $S$, that intersects $\Lambda$ trivially, $\widetilde{S}$ is canonically isomorphic to $K^{n+1}/\widetilde{\Lambda}$, and so we see that $T_\Lambda \mathbb{G} = \mathrm{Hom}(\widetilde{\Lambda}, K^{n+1}/\widetilde{\Lambda})$.

If we want to be still more explicit, let $C = \{\Lambda(t)\} \subset \mathbb{G}$ is a curve in the Grassmannian parametrized by $t$ such that $\Lambda(0) = \Lambda$. Furthermore, suppose $C$ is smooth at $\Lambda$. This means that $T_\Lambda C$ is a one dimensional vector subspace of $\mathrm{Hom}(\widetilde{\Lambda}, K^{n+1}/\widetilde{\Lambda})$. Let $\varphi \in T_\Lambda C$ be a nonzero vector. If we choose some $v \in \Lambda$, we must determine the action of $\varphi$ on $v$. To do this, choose any curve $C' = \{v(t)\} \subset \mathbb{P}^n$ such that $v(t) \in \Lambda(t)$ and $v(0) = v$. Then because $\varphi$ is a tangent vector to $C$, it must map $v$ to the tangent vector of $C'$. So $\varphi(v) = v'(0)$. We now show that $\varphi$ does not depend on the choice of the curve $C'$. Let $C'' = \{w(t)\} \subset \mathbb{P}^n$ be another curve such that $w(t) \in \Lambda(t)$ for every $t$ and $w(0) = v(0) = v$. Let $\{\widetilde{v}(t)\} \subset \mathbb{A}^{n+1}$ be a curve lying above $C'$. Similarly, let $\{\widetilde{w}(t)\} \subset \mathbb{A}^{n+1}$ be a curve lying above $C''$. Then $\widetilde{w}(t) - \widetilde{v}(t) \in \widetilde{\Lambda}(t)$ for every $t$, and so

$$\widetilde{u}(t) = \frac{\widetilde{w}(t) - \widetilde{v}(t)}{t} \in \widetilde{\Lambda}(t),$$

for every nonzero $t$. As it stands, $\{\widetilde{u}(t)\}$ is not closed, however, if we correct this by defining

$$\widetilde{u}(0) = \widetilde{w'}(0) - \widetilde{v}'(0).$$

Since $\widetilde{u}(t) \in \widetilde{\Lambda}(t)$ for every $t$, we see that $\varphi$ is well defined as a map to $K^{n+1}/\widetilde{\Lambda}$.

Finally, note that two curves in the Grassmannian, $C = \{\Lambda(t)\}$ and $D = \{\Gamma(t)\}$ satisfying $\Lambda(0) = \Gamma(0) = 0$ will give rise to the same tangent vector $\varphi$ if and only if curves $\{v(t)\}$ and $\{w(t)\}$ can be chosen satisfying $v(t) \in \Lambda(t)$, $w(t) \in \Gamma(t)$ and $v(0) = w(0) = v$ so that $\{v(t)\}$ and $\{w(t)\}$ are tangent at $v$, which is possible if and only if the Grassmannian curves $C$ and $D$ are tangent at $\Lambda$. Finally, given any linear map $\varphi \in \mathrm{Hom}\big(\widetilde{\Lambda}, K^{n+1}/\widetilde{\Lambda}\big)$, we can construct a curve, $C$, in the Grassmannian so that $\varphi$ arises as the tangent vector to $C$ at $\Lambda$. Therefore, we have a natural identification of $T_\Lambda\mathbb{G}$ with $\mathrm{Hom}\big(\widetilde{\Lambda}, K^{n+1}/\widetilde{\Lambda}\big)$.

## 5.2 The Second Fundamental Form - A First Glance

There are many ways to understand the second fundamental form of a variety $X$ at a point $p$. In differential geometry, for example, the second fundamental form of a real manifold at a point may be written down explicitly as a symmetric bilinear form on the tangent space which can be used to quantify the curvature of the manifold. We do not need to be quite so explicit for our algebraic purposes, which is lucky because it might have been difficult to use a value in an arbitrary characteristic 0 field $K$ to quantify anything. Nevertheless, the intuition behind the second fundamental form is the same. In short, it measures the motion of the tangent space away from a given vector along another given vector. We develop the second fundamental form here in two somewhat different ways. We begin with the more classical approach using the Gauss map, and then, because it will make several of the properties we will need more clear, we develop it by looking at the Taylor expansion of a tangent hyperplane section. In order to define the second fundamental form, we must first define the Gauss map. If $X \subset \mathbb{P}^n$ is a $k$-dimensional variety, then the Gauss map of $X$ is a map

$$\begin{aligned} \mathcal{G}: \quad & X \dashrightarrow \mathbb{G}(k,n) \\ & p \mapsto \mathbb{T}_pX. \end{aligned}$$

**Note.** Clearly, $\mathcal{G}$ is regular on $X^{\mathrm{sm}}$.

The differential of the Gauss map, therefore, is a map

$$(d\mathcal{G})_p : T_pX \to \mathrm{Hom}(\widetilde{\Lambda}, K^{n+1}/\widetilde{\Lambda}),$$

where $\Lambda = \mathbb{T}_pX$. In order to be more explicit, let

$$\varphi_i = (d\mathcal{G})_p\left(\frac{\partial v}{\partial t_i}(0)\right).$$

Then, by reasoning similar to that at the end of the discussion of the tangent spaces to the Grassmannian section, we see that $\varphi_i$ will act on $\widetilde{\Lambda}$ by

$$\varphi_i\left(\frac{\partial v}{\partial t_j}(0)\right) = \frac{\partial^2 v}{\partial t_i \partial t_j}(0),$$

and

$$\varphi_i(p) = \frac{\partial v}{\partial t_i}(0).$$

Clearly then, $p \in \ker \varphi_i$ for every $i$, and so we have a map

$$(d\mathcal{G})_p : T_pX \to \mathrm{Hom}(\widetilde{\Lambda}/\widetilde{p}, K^{n+1}/\widetilde{\Lambda}) = \mathrm{Hom}(T_pX, N_pX).$$

This map may also be viewed as

$$(d\mathcal{G})_p : T_pX \otimes T_pX \to N_pX.$$

**Note.** This last map is clearly symmetric in its arguments because

$$\frac{\partial^2 v}{\partial t_i \partial t_j}(0) = \frac{\partial^2 v}{\partial t_j \partial t_i}(0).$$

Therefore, we have a map

$$(d\mathcal{G})_p : \mathrm{Sym}^2(T_pX) \to N_pX,$$

which we can dualize to obtain a map

$$(d\mathcal{G})_p^* : N_pX^* \to \mathrm{Sym}(T_pX^*).$$

The last observation we must make before defining the second fundamental form is that the set of symmetric bilinear forms on $T_pX$ is naturally identified with the set of quadrics on $T_pX$. Explicitly, this map sends a linear form on $N_pX$, say $\varphi$, to a quadratic polynomial on $T_pX$, say $\psi$ where

$$\psi : a_1 \frac{\partial v}{\partial t_1}(0) + \cdots + a_k \frac{\partial v}{\partial t_k}(0) \mapsto \sum_{i,j=1}^{n} a_i a_j \cdot \varphi\left(\frac{\partial^2 v}{\partial t_i \partial t_j}(0)\right).$$

**Definition.** We define the *second fundamental form of $X$ at $p$*, denoted $II_p$, to be the quadrics on $T_pX$ contained in the image of $(d\mathcal{G})_p^*$.

**Note.** Because the map $(d\mathcal{G})_p^*$ is linear, the quadrics in the image can be seen to span a vector space. Therefore, the second fundamental form is a linear system of quadrics. In true projective spirit, we say that the dimension of the second fundamental form, denoted $\dim II_p$, is one less than the dimension of this vector space of quadrics.

**Definition.** We define the *base locus* of the second fundamental form of $X$ at $p$, denoted $\mathcal{B}_p$ to be the common zero locus of the quadrics in $II_p$.

**Note.** As we have defined it, the base locus is a subvariety of the Zariski tangent space, $T_pX$, however, we generally prefer to view it as a projective subvariety of $\mathbb{T}_pX$ by taking the closure of the embedding $T_pX \hookrightarrow \mathbb{T}_pX$. When there is danger of confusion, we will call the subvariety of the Zariski tangent space the *Zariski base locus*, and we will call the subvariety of the projective tangent space the *projective base locus*.

**Note.** Since all of the quadratic polynomials in $II_p$ are homogeneous, the Zariski base locus will be a cone with the origin as its vertex. This means that the projective base locus is a cone with vertex $p$.

We make one observation before describing an alternative, but equivalent viewpoint of the second fundamental form.

**Claim.** If $\Lambda \in F_r(X)$ is an $r$-plane that passes through the smooth point $p$ then $\Lambda \subset \mathcal{B}_p$.

*Proof.* If we choose local parameters for $X$ around $p$,

$$v(t_1, \ldots, t_k) = v(T) = [v_0(T), \ldots, v_n(T)],$$

then we can assume that $v$ is linear with respect to $t_1, \ldots, t_r$ (since $\Lambda \subset X$). Therefore,

$$\frac{\partial^2 v}{\partial t_i \partial t_j} = 0,$$

for every $1 \leq i, j \leq r$. Since the points of $\mathbb{T}_p X$ corresponding to $\Lambda$ are spanned by $\left\{p, \frac{\partial v}{\partial t_i}(0) : i = 1, \ldots, r\right\}$, every point of $\Lambda$ will be in the zero locus of every quadric in $II_p$ and so $\Lambda \subset \mathcal{B}_p$. $\square$

## 5.3 The Second Fundamental Form - Another First Glance

Again, we let $X \subset \mathbb{P}^n$ be a $k$-dimensional variety locally parametrized by $v(t_1, \ldots, t_k)$ with $v(0) = p$, a smooth point. Then for any hyperplane containing $p$, $H \subset \mathbb{P}^n$ defined by the linear form $\psi_H(S_0, \ldots, S_n) = 0$, we have that $H \cap X$ is locally defined by

$$\psi_H\big(v(t_1, \ldots, t_k)\big) = 0,$$

which we Taylor expand, to get

$$0 = \psi_H(p) + \sum_{i=1}^{n} \psi_H\left(\frac{\partial v}{\partial t_i}(0)\right) t_i + \frac{1}{2} \sum_{i,j=1}^{n} \psi_H\left(\frac{\partial^2 v}{\partial t_i \partial t_j}(0)\right) t_i t_j + \cdots .$$

Clearly $\psi_H(p) = 0$ since $p \in H$. Additionally, if we let $H$ be a tangent hyperplane, then the second sum will also vanish. Therefore, when $H$ is a tangent hyperplane, we have

$$0 = \psi_H\left(\frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 v}{\partial t_i \partial t_j}(0) t_i t_j\right) + \text{(higher order terms)},$$

and so we get a map from the set of all tangent hyperplanes to the set of quadratic polynomials on $\mathbb{A}^k \simeq T_p X$ (since $\mathbb{A}^k$ and $T_p X$ are $K$-vector spaces of the same dimension). Again, since quadratic polynomials may be seen as symmetric bilinear forms, we have a map from tangent hyperplanes to $\mathrm{Sym}^2(T_p X^*)$. Finally, note that the set of hyperplanes containing $\Lambda$ is naturally identified with $N_p X^*$, since a hyperplane containing $\Lambda$ is the same thing as a linear form on $K^{n+1}$ that vanishes on $\widetilde{\Lambda}$ (recall $\widetilde{\Lambda}$ is the $(k+1)$-plane in $\mathbb{A}^{n+1}$ lying above $\Lambda$). Therefore, we have a map

$$N_p X^* \to \mathrm{Sym}^2\big(T_p X^*\big) : \psi_H \mapsto \varphi_H,$$

where

$$\varphi_H\left(a_1 \frac{\partial v}{\partial t_1}(0) + \cdots + a_k \frac{\partial v}{\partial t_k}(0)\right) = \frac{1}{2} \sum_{i,j=1}^{n} a_i a_j \cdot \psi_H\left(\frac{\partial^2 v}{\partial t_i \partial t_j}(0)\right).$$

Therefore, our map $N_pX^* \to \mathrm{Sym}^2(T_pX^*)$ is a scalar times the map $(d\mathcal{G})_p^*$ from the previous section. In particular, the linear system of quadrics in the image will have the same dimension, and the same set of common zeros. Therefore, the second fundamental form may be defined in terms of the map obtained in this way as well. By developing the second fundamental in this way, it is immediately clear that the base locus, $\mathcal{B}_p$, is the union of the lines in $\mathbb{T}_pX$ that intersect $X$ at $p$ with multiplicity at least 3. This observation gives the following useful result.

**Claim.** If $X \subset \mathbb{P}^n$ is an irreducible $k$-dimensional variety other than a $k$-plane, then the intersection multiplicity of $\mathbb{T}_pX$ and $X$ at $p$ is two, for general $p \in X$.

*Proof.* Clearly any variety $X \subset \mathbb{P}^n$ intersects its tangent plane $\mathbb{T}_pX$ tangentially at $p$, and so the multiplicity of intersection is at least two.

Now, if $p \in X$ is such that the intersection multiplicity of $\mathbb{T}_pX$ and $X$ at $p$ is greater than two, then any line through $p$ in $\mathbb{T}_pX$ also intersects $X$ at $p$ with multiplicity greater than two. This tells us that the base locus of $II_p$ is all of $\mathbb{T}_pX$, and so it must be that

$$\frac{\partial^2 v}{\partial t_i \partial t_j}(0) = 0$$

for every $i$ and $j$. Now, if the general point of $X$ has the property that the intersection of $X$ with its tangent plane has multiplicity at least 3, then it must be that

$$\frac{\partial^2 v}{\partial t_i \partial t_j}(x_1, \ldots, x_k) = 0$$

for general $(x_1, \ldots, x_k) \in T_pX$. This tells us that

$$\frac{\partial^2 v}{\partial t_i \partial t_j} \equiv 0$$

for every $i$ and $j$, which says exactly that the local parameters of $X$ are linear, and so $X$ is a $k$-plane. $\square$

## 5.4 Two Useful Results

In this section we cite two basic theorems regarding the second fundamental form which we will use periodically throughout the remainder of the paper. For a proof of the following, see [T].

**Proposition 1.** For $X \subset \mathbb{P}^n$ a projective variety and $p \in X$ any point, let

$$\sigma_p = k - \dim II_p - 1.$$

Then there exists a variety $Z = \infty^h \mathbb{P}^m$ such that $X \subset Z$ and the tangent planes to $Z$ at smooth points of the general $\mathbb{P}^m$ are contained in a fixed $\mathbb{P}^{2k-h-\sigma_p}$ for some $h = 0, \ldots, k - \sigma_p$.

The following result is proved in [G-H]

**Proposition 2.** Let $X \subset \mathbb{P}^n$ be a $k$-dimensional variety, and let $p \in X$ be a general point. Suppose that the quadrics of $II_p$ have a fixed hyperplane in common. Furthermore, suppose that $\dim II_p \geq 1$. Then $X$ is either a one-parameter family of $(k-1)$-planes; or else it is a two-parameter family of $(k-2)$-planes.

# 6 Another Classification Theorem

## 6.1 A Maximum for $\dim F_r(X)$ when $X$ is not a Plane

**Theorem 2.** If $X \subset \mathbb{P}^n$ is a $k$-dimensional variety other than a $k$-plane, then $\dim F_r(X) \leq (r+1)(k-r) - r$.

*Proof.* When $r = 1$ this follows from theorem 1. Now let $X \subset \mathbb{P}^n$ be a $k$-dimensional variety other than a $k$-plane such that $\dim F_r(X) \geq (r+1)(k-r)-r$. We will show that equality holds. Notice that $X$ is still swept out by its Fano variety because any proper subvariety, $X' \subset X$, must have $\dim F_r(X') \leq (r+1)(k-r)$ by the upper bound for $\dim F_r(X')$. Choose a general hyperplane $H \subset \mathbb{P}^n$, and, as before, we study the map

$$\varphi_H : F_r(X) \dashrightarrow F_{r-1}(X \cap H).$$

It suffices to show that for general $\Gamma \in \mathrm{Im}\varphi_H$, $\dim \varphi_H^{-1}(\Gamma) \leq k - r - 1$, because then the theorem will follow by induction. Note that since $X$ is swept out by its Fano variety, and $H \subset \mathbb{P}^n$ and $\Gamma \in \mathrm{Im}\varphi_H$ are general, a general point $p \in \varphi_H^{-1}(\Gamma)$ is a general point of $X$. Let $p$ be such a point. As we saw previously,

$$\varphi_H^{-1}(\Gamma) \subset \mathbb{G}_\Gamma,$$

where $\mathbb{G}_\Gamma$ is the set of $r$-planes in $\mathbb{T}_pX = \mathbb{P}^k$ containing the $(r-1)$-plane $\Gamma$. Notice that $\varphi_H^{-1}(\Gamma)$ cannot be dense in $\mathbb{G}_\Gamma$ since this would mean that $X$ would contain $\mathbb{T}_pX$ (since for any point $x \in \mathbb{T}_pX$, the $r$-plane spanned by $\Gamma$ and $x$ must be in $F_r(X)$ and so $x \in X$). So since $p \in X$ was general, $X$ would have to contain its general tangent plane, and so $X$ would have to be a plane, contrary to our hypothesis. We see, therefore, that $\dim \varphi_H^{-1}(\Gamma) \leq k-r-1$, as desired. $\square$

## 6.2 One-Parameter Families of $(k-1)$-planes

One type of $k$-dimensional variety which we would expect to contain many lines is a one-parameter family of $\mathbb{P}^{k-1}$'s. Formally, such a variety is defined as

$$X = \bigcup_{\Lambda \in C} \Lambda \subset \mathbb{P}^n,$$

where $C \subset \mathbb{G}(k-1, n)$ is a Grassmannian curve, and where the general point $p \in X$ lies in exactly one $(k-1)$-plane of $C$. Clearly $X$ is a variety, as it is the image of the incidence correspondence

$$Z = \{(x, \Lambda) : x \in \Lambda\} \subset \mathbb{P}^n \times C$$

under projection onto the first component. Furthermore, we use the theorem of the fibers on the projection maps to deduce that $\dim X = k$.

Notice that

$$\bigcup_{\Lambda \in C} F_r(\Lambda) \subset F_r(X),$$

and since

$$\dim\Bigg(\bigcup_{\Lambda\in C} F_r(\Lambda)\Bigg) = (r+1)(k-r) - r,$$

it must be a maximal component of $F_r(X)$.

**Note.** In order to show that a variety $X$ is a one-parameter family of $\mathbb{P}^{k-1}$, it suffices to show that the general point is contained in a $\mathbb{P}^{k-1}$ that is containd in $X$, because with this information we can set up an incidence correspondence and project into $\mathbb{G}(k-1,n)$ to obtain a curve parametrizing $X$ (note even when a general point is contained in a positive dimensional family of $\mathbb{P}^{k-1}$, we can set up the incidence correspondence and project to $\mathbb{G}(k-1,n)$ to obtain a variety of dimension say $m > 1$ which we can then intersect with a general $(N-m+1)$-plane in $\mathbb{P}(\wedge^k \mathbb{A}^n)$ to obtain a curve parametrizing $X$). Furthermore, in order to show that a $k$-dimensional variety is an $h$-dimensional family of any type of variety (quadrics, for example) we simply need to show that a general point is contained in a $(k-h)$-dimensional variety of the proper type.

Finally, some terminology.

**Definition.** We say that the variety swept out by a family of $\mathbb{P}^m$ is a *scroll in* $\mathbb{P}^m$.

## 6.3 Quadrics

Quadric hypersurfaces form another class of variey that is similar to a plane. Indeed, their degree is as close to that of a plane as possible. It is not unreasonable, therefore, to suspect that they will contain many planes.

Let $\mathbb{P}^N$ be the projective space parametrizing the set of quadric hypersurfaces in $\mathbb{P}^n$. Define the variety

$$Z = \{(\Lambda, X) : \Lambda \subset X\} \subset \mathbb{G}(r,n) \times \mathbb{P}^N,$$

and let $\pi_1 : Z \to \mathbb{G}(r,n)$ and $\pi_2 : Z \to \mathbb{P}^N$ be the projections. Clearly $\pi_1$ is surjective. Also, note that for any $\Lambda \in \mathbb{G}(r,n)$, $\pi_1^{-1}(\Lambda)$ is the set of quadrics vanishing on $\Lambda$. If we choose coordinates $S_0, \ldots, S_n$ for $\mathbb{P}^n$ so that $\Lambda = \{S_{r+1} = \cdots = S_n = 0\}$, then we see that $\pi_1^{-1}(\Lambda)$ is the set of quadrics where every monomial is divisible by $S_i$ for some $i = r+1, \ldots, n$. The number of monomials, theerefore, which are nonzero on $\Lambda$ is the number of ways to choose $(r+1)$ nonzero integers that add to 2. Combinatorics gives us

$$\dim \pi_1^{-1}(\Lambda) = N - \binom{r+2}{2}.$$

Finally, we show that when $2r < n$, $\pi_2$ is surjective. Since the general quadric hypersurface is smooth, and all smooth quadric hypersurfaces are isomorphic, we simply must exhibbit a single smooth quadric containing an $r$-plane. This is straightforward. Let $X \subset \mathbb{P}^n$ be the quadric hypersurface defined by

$$S_0 S_n + S_1 S_{n-1} + \cdots + S_{\lfloor n/2 \rfloor} S_{\lceil n/2 \rceil} = 0.$$

Then $X$ is smooth, and will contain the $r$-plane

$$\Lambda = \{S_{r+1} = \cdots = S_n = 0\}.$$

**Note.** We will see later that if $X \subset \mathbb{P}^n$ is a hypersurface containing an $r$-plane for some $2r \geq n$, then $X$ must be singular. In particular, when $2r \geq n$, $\pi_2$ is not surjective.

The theorem of the fibers gives us that for general quadric hypersurfaces $X$,

$$\dim F_r(X) = \dim \mathbb{G}(r, n) - \binom{r+2}{2} = (r+1)(n-1-3r/2).$$

In particular, note that when $r = 1$, $\dim F_r(X) = 2(n-1) - 3$, which is the maximum that it could be since quadrics are not planes. Our next classification theorem says that if $X \subset \mathbb{P}^n$ is a $k$-dimensional variety with $\dim F_r(X) = (r+1)(k-r) - r$ then $X$ is a scroll in $\mathbb{P}^{k-1}$ or $r = 1$ and $X$ is a quadric hypersurface.

## 6.4 Proof of the Classification Theorem

In this section we prove the following theorem.

**Theorem 3.** If $X \subset \mathbb{P}^n$ is an irreducible $k$-dimensional variety with $\dim F_r(X) = (r+1)(k-r) - r$, then $X$ is a scroll in $\mathbb{P}^{k-1}$, or $r = 1$ and $X$ is a quadric hypersurface.

*Proof.* We break the proof into several steps.

**Step 1:** $r = 1$ **(1)**

In this section, we prove that if $X \subset \mathbb{P}^n$ is a $k$-dimensional variety with $\dim F_1(X) = 2k - 3$ then either $X$ is a scroll in $\mathbb{P}^{k-1}$ or $n = k+1$ and $X$ is a hypersurface.

Let $X \subset \mathbb{P}^n$ be a $k$-dimensional variety and let $\Sigma = F_1(X)$. Suppose that $\dim \Sigma = 2k - 3$. By examining the variety

$$Z = \{(p, l) : p \in l\} \subset X \times \Sigma$$

we see that through the general point $p \in X$ there passes a $(k-2)$-dimensional family of lines in $\Sigma$, denoted $\Sigma_p$. Just as in the proof of theorem 1, $\Sigma_p$ sweeps out a $(k-1)$-dimensional variety, $X_p \subset X$. Since $X_p$ is swept out by lines in $X$ through $p$, $X_p \subset \mathcal{B}_p$. Therefore, $\dim \mathcal{B}_p = k - 1$ (since $X$ is not a plane).

Now let $\{Q_1, \ldots, Q_N\}$ be quadratic polynomials which span $II_p$. If $N \geq 2$ then each $Q_i$ must be reducible, and they all must share a factor (since their common zero locus is $(k-1)$-dimensional). Therefore, if $N \geq 2$, then $\dim II_p \geq 1$ and the quadrics of the second fundamental form share a fixed hyperplane. By proposition 2, we have either $X = \infty^2\mathbb{P}^{k-2}$, or $X = \infty^1\mathbb{P}^{k-1}$. In the first case, $\dim F_1(X) = 2k-4$, and so if $\dim II_p \geq 1$ $X = \infty^1\mathbb{P}^{k-1}$.

If $\dim II_p = 0$ then we show that either $X = \infty^1\mathbb{P}^{k-1}$ or $X$ is a hypersurface by using proposition 1. If $\dim II_p = 0$ then in the notation of proposition 1, $\sigma_p = k-1$ and so we get that $X \subset Z = \infty^h\mathbb{P}^m$ where the family of tangent $\mathbb{P}^{h+m}$ at the general smooth points of a general $\mathbb{P}^m$ lie in a fixed $\mathbb{P}^{k-h+1}$. This gives us

$$k \leq m + h \leq k - h + 1,$$

and so $h = 0$ or $h = 1$. If $h = 0$, then $m \leq k+1$ and so $X$ is a hypersurface as we have $X \subset \mathbb{P}^{k+1}$. If $h = 1$ then $m = k-1$ and $X = \infty^1\mathbb{P}^{k-1}$.

**Step 2:** $r = 1$ **(2)**

In this section, we prove the theorem for the case of $r = 1$. By the above, all we must show is that if $X \subset \mathbb{P}^n$ is a hypersurface such that $\dim F_1(X) = 2k-3$, then $\deg X = 2$. To begin with, we suppose that $k = 2$. So $X \subset \mathbb{P}^3$ is a surface containing a one-dimensional family of lines. Recall from the second fundamental form section that for general $p \in X$, $\mathbb{T}_pX$ and $X$ intersect at $p$ with multiplicity 2. Therefore, $p$ is a singular point of order 2 of the curve $\mathbb{T}_pX \cap X$. Since the finitely many lines in $X$ through $p$ (recall from step 1, $\dim \Sigma_p = k-2$) are contained in $\mathbb{T}_pX$, it must be that they are each irreducible components of $\mathbb{T}_pX \cap X$. Since $p$ is a double point, it will be contained in exactly two (not necessarily distinct since we allow for multiplicity) components of $\mathbb{T}_pX \cap X$. Therefore, either there are two lines in $X$ passing through $p$ or there is one double line. In either case,

$$\deg X = (\deg X)(\deg \mathbb{T}_pX) = 2,$$

as desired.

When $X$ is a $k$-dimensional hypersurface we simply intersect $X$ with a general $(n-k+2)$-plane to obtain a surface in $\mathbb{P}^3$. Note that if $X$ was such that $\dim F_1(X) = 2k-3$, then the general $(n-k+2)$-plane $\Gamma \subset \mathbb{P}^n$ will be such that $X \cap \Gamma$ will be ruled. To see this, it suffices to show that if $\dim F_1(X) = 2k-3$ then for a general hyperplane $H \subset \mathbb{P}^n$, $\dim F_1(X \cap H) = 2k-5$. This is immediate upon examining the variety

$$Z = \{(l, H) : l \subset H\} \subset F_1(X) \times \mathbb{G}(k, k+1)$$

and its projections. We get that for general $l$ and $H$,

$$\dim F_1(X \cap H) = \dim F_1(X) + \dim \mathbb{G}_l - \pi_2(Z) \geq 2k-5.$$

Therefore, if $X \subset \mathbb{P}^{k+1}$ is such that $\dim F_1(X) = 2k-3$ then for the general $(n-k+2)$-plane $\Gamma \subset \mathbb{P}^{k+1}$, $X \cap \Gamma \subset \mathbb{P}^3$ is a quadric hypersurface. Since $\Gamma$ was general it intersects $X$ transversely and so we have

$$\deg X = \deg(X \cap \Gamma) = 2,$$

which completes the theorem for $r = 1$.

**Step 3:** $r > 1$

We must show that the only varieties satisfying $\dim F_r(X) = (r+1)(k-r) - r$ are scrolls in $\mathbb{P}^{k-1}$. Just as in the proof of theorem 1 we choose a general hyperplane $H \subset \mathbb{P}^n$ and we define the map

$$\begin{aligned} \varphi_H : \quad & F_r(X) \dashrightarrow F_{r-1}(X \cap H) \\ & \Lambda \mapsto \Lambda \cap H. \end{aligned}$$

The theorem of the fibers gives us

$$\begin{aligned} \dim F_{r-1}(X \cap H) & \geq \dim F_r(X) - \dim \varphi_H^{-1}(\Gamma) \\ & \geq (r+1)(k-r) - r - (k-r-1) \\ & = r(k-r) - (r-1), \end{aligned}$$

where the second line is due to the same argument as in the proof of theorem 2; namely, $\varphi_H^{-1}(\Gamma) \subset \mathbb{G}_\Gamma$ where $\mathbb{G}$ is the set of $r$-planes in the tangent space to $X$ at a point contained in $\Gamma$. Moreover, it is a proper subvariety because otherwise $X$ would be a plane.

Therefore, we see that if $X \subset \mathbb{P}^n$ is a $k$-dimensional variety with $\dim F_1(X) = 2k-3$, then for a general $(n-r+1)$-plane $\Gamma \subset \mathbb{P}^n$, $\dim F_1(X \cap \Gamma) = 2k-3$. Therefore, $X \cap \Gamma$ is either a scroll or a quadric hypersurface, which means, since $\Gamma$ was general, that $X$ is either a scroll or a quadric hypersurface. However, by the same argument we have used twice, we can see that when $X$ is a quadric hypersurface, $\dim \varphi_H^{-1}(\Gamma) \leq k-r-2$, and so when $r > 1$, the only varieties with $\dim F_r(X) = (r+1)(k-r) - r$ are one-parameter families of $\mathbb{P}^{k-1}$, as desired. $\square$

# 7 One More Classification Theorem

**Definition.** If $X \subset \mathbb{P}^n$ is an irreducible, $k$-dimensional variety such that $\dim F_1(X) = 2k - 2 - N$, and furthermore $X$ is swept out by its Fano variety, then we say that $X$ is of *type* $R_N$.

In theorem 1 and theorem 3 we classify varieties of type $R_0$ and $R_1$. A reasonable next question for one to ask is 'can we give a classification of type $R_2$ varieties?' In this section we will give a complete classification in the case that $X \subset \mathbb{P}^n$ has codimension greater than 2. We first aquaint ourselves with some type $R_2$ varieties.

## 7.1 Standard Varieties

**Definition.** We say that a variety is *standard* if it is a one-parameter family of varieties of type $R_{N-1}$.

**Note.** If $X$ is such a variety, then $X$ will be of type $R_N$, since its Fano variety of lines will contain

$$\bigcup_{x \in C} F_1(X_x),$$

where $C$ is a curve parametrizing $X$ and $\dim X_x = \dim X - 1$ for every $x \in C$. Furthermore, its Fano variety cannot contain a component of greater dimension since then a one parameter family of type $R_{N-1}$ varieties would be of type $R_{N-1}$, which is not the case for general families.

We see, therefore, that two-parameter families of $\mathbb{P}^{k-2}$ and one-parameter families of quadrics will be of type $R_2$. We might expect that these will be all of they type $R_2$ varieties with codimension at least 3, however we have another surprising example.

## 7.2 The Grassmannian $\mathbb{G}(1, 4)$

Let $\mathbb{G} = \mathbb{G}(1, 4)$. Recall that $\mathbb{G} \subset \mathbb{P}^9$, and also that $\dim \mathbb{G} = 6$. In order to verify that $\mathbb{G}$ is type $R_2$, therefore, we must check that $\dim F_1(\mathbb{G}) = 8$. It is not immediately clear, however, what a one-dimensional linear subvariety of the Grassmannian looks like. A natural guess might be they are sets of lines which sweep out a plane in $\mathbb{P}^4$. This is not quite right, however, as the following example shows.

**Example.** Choose any 2-plane, $\Gamma \subset \mathbb{P}^4$, and let $C \subset \Gamma$ be a smooth curve of degree 2. Let

$$\mathcal{G} : C \to \mathbb{G}(1, 2) \subset \mathbb{G}(1, 4)$$

be the Gauss map, and let $Y = \mathcal{G}(C) \subset \mathbb{G}(1, 4)$. Since the fibers of the Gauss map are finite, the variety swept out by the lines of $Y$ will be two-dimensional, and will be contained in $\Gamma$. Therefore, $Y$ sweeps out $\Gamma$. We show, however, that $Y \subset \mathbb{G}(1, 4)$ is not a line by showing that $\deg Y \geq 2$.

Since $\mathbb{G}(1,2) \simeq \mathbb{P}^2$, we can view the Gauss map as a map $\mathcal{G} : C \to \mathbb{P}^2$. Therefore, to calculate $\deg_0(\mathcal{G})$, we must find the number of points in the preimage of a general line in $\mathbb{G}(1,2)$. Choose general points $p_1, p_2, p_3 \in \Gamma$ (note that the $p_i$ span $\Gamma$). If we choose vectors $v_i \in \widetilde{p}_i$ (where $\widetilde{p}_i$ is the line in affine 3-space lying over $p_i$), then the points of $\mathbb{G}(1,2)$ are linear combinations in the $v_i \wedge v_j$. Therefore,

$$L = \{A[v_1 \wedge v_2] + B[v_1 \wedge v_3] : [A,B] \in \mathbb{P}^1\}$$

is a general line in $\mathbb{G}(1,2)$ (that $L \subset \mathbb{G}(1,2)$ is clear because $\mathbb{G}(1,2) \simeq \mathbb{P}^2$). Notice that

$$\mathcal{G}^{-1}(L) = \{p : p_1 \in \mathbb{T}_p C\}.$$

For a fixed line $l \in Y$, define the map

$$\begin{aligned} \varphi_l : \quad & C \dashrightarrow \mathbb{P}^2 \\ & p \mapsto l \cap \mathbb{T}_p C. \end{aligned}$$

Note that $\varphi_l$ is regular away from the point of intersection between $l$ and $C$. Clearly $\dim \varphi_l(C) = 1$ since otherwise every tangent line of $C$ would pass through a single point of $\mathbb{P}^2$, which clearly cannot be the case (we could, for example, explicitly write down the equation for $C$ and get the equation for $\mathbb{T}_p C$ to show directly that there is no such point). This means that for general $x \in \mathbb{P}^2$, and some line $l \in Y$ containing $x$, $x$ will be the intersectiton of $l$ and $l'$ for some $l' \in Y$. Therefore, a general point is contained in at least two lines in $Y$ and so $\deg_0(\mathcal{G}) \geq 2$.

We have

$$(\deg Y)(\deg \mathcal{G}) = \deg_0(\mathcal{G}) \geq 2.$$

But $\deg \mathcal{G} = 1$ since any line in $Y$ will be tangent to $C$ at exactly one point because $\deg C = 2$ and so if a line intersects $C$ tangently at a point, it will not intersect $C$ again (tangentially or otherwise). Therefore, we see that $\deg Y \geq 2$ and so even though the lines of $Y$ sweep out $\Gamma$, $Y$ is not a linear subspace of $\mathbb{G}(1,4)$.

The previous example, however, suggests a guess as to what the lines in $\mathbb{G}(1,4)$ might be. We saw that the lines of $\mathbb{G}(1,2)$ were spanned by the 2-forms $v_1 \wedge v_2$ and $v_1 \wedge v_3$, and the following claim tells us that the lines in $\mathbb{G}(1,4)$ will have a similar form. Now that $\mathbb{G}(1,2)$ is out of the picture, we are in no danger of confusing the two different Grassmannians, and so we return to letting $\mathbb{G} = \mathbb{G}(1,4)$.

**Claim.** The lines in $\mathbb{G}$ are exactly the subsets of the form

$$L = \{A[v_1 \wedge v_2] + B[v_1 \wedge v_3] : [A,B] \in \mathbb{P}^1\},$$

for three linearly independent vectors $v_1, v_2, v_3 \in \mathbb{A}^5$.

*Proof.* First, note that any such set is a line of $\mathbb{G}$ since it is the span of two points of $\mathbb{G}$ and

$$A\big[v_1 \wedge v_2\big] + B\big[v_1 \wedge v_3\big] = \big[v_1 \wedge (Av_2 + Bv_3)\big] \in \mathbb{G}$$

for every $[A, B] \in \mathbb{P}^1$.

Next, if $L' \subset \mathbb{G}$ is any line, it must be spanned by two points of $\mathbb{G}$, say $v_1 \wedge v_2$ and $v_3 \wedge v_4$. If the $v_i$ are linearly independent, then $v_1 \wedge v_2$ and $v_3 \wedge v_4$ are basis vectors of $\wedge^2 \mathbb{A}^5$, and so $v_1 \wedge v_2 + v_3 \wedge v_4$ cannot be writtin as the wedge product of two vectors, and so we must have

$$v_4 = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3.$$

Therefore,

$$\begin{aligned}
Av_1 \wedge v_2 + Bv_3 \wedge v_4 &= Av_1 \wedge v_2 + Bv_3 \wedge (\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3) \\
&= Av_1 \wedge v_2 - B(\alpha_1 v_1 + \alpha_2 v_2) \wedge v_3 \\
&= \frac{A}{\alpha_1}(\alpha_1 v_1 \wedge v_2) - B(\alpha_1 v_1 + \alpha_2 v_2) \wedge v_3 \\
&= \frac{A}{\alpha_1}(\alpha_1 v_1 + \alpha_2 v_2) \wedge v_2 - B(\alpha_1 v_1 + \alpha_2 v_2) \wedge v_3 \\
&= \big(\alpha_1 v_1 + \alpha_2 v_2\big) \wedge \big(\frac{A}{\alpha_1} v_2 - Bv_3\big),
\end{aligned}$$

where we assume $\alpha_1 \neq 0$ because otherwise we are done. When we let

$$v_1' = \frac{1}{\alpha_1}(\alpha_1 v_1 + \alpha_2 v_2);\ \ v_2' = v_2;\ \text{ and } v_3' = -\alpha_1 v_3,$$

we get

$$Av_1 \wedge v_2 + Bv_3 \wedge v_4 = Av_1' \wedge v_2' + Bv_1' \wedge v_3',$$

and so the claim is proved. ☐

**Note.** The above claim tells us that the one-dimensional linear subvarieties of $\mathbb{G}$ are exactly the sets of lines through a point $(v_1)$ and contained in a 2-plane $(\mathrm{Span}\{v_1, v_2, v_3\})$.

**Note.** If we had thought a bit harder before making our first guess (that lines in $\mathbb{G}$ are families of lines which sweep out a plane), we would have come up with this, since a curved family of lines (such as the family in our example) can still sweep out a plane, but it shouldn't be that a curved family of lines should correspond to a line. It should be that only noncurved families of lines correspond to lines. A set of lines in a $\mathbb{P}^2$ containing a point can be seen to be a noncurved family by letting the point in common be a point at $\infty$ in $\mathbb{P}^2$ which would make the family of lines in the affine chart look like a family of parallel lines.

This last observation allows us to compute $\dim F_1(\mathbb{G})$. It is simply the dimension of the set of pairs

$$Z = \{(p, \Gamma) : p \in \Gamma\} \subset \mathbb{P}^4 \times \mathbb{G}(2, 4).$$

We use the theorem of the fibers on the projection onto the second component to see that $\dim F_1(\mathbb{G}) = \dim Z = 8$, and so $\mathbb{G}$ is of type $R_2$.

## 7.3 The Theorem

The partial classification theorem for varieties of type $R_2$ that we will prove is the following.

**Theorem 4.** Let $X \subset \mathbb{P}^n$ be a $k$-dimensional, type $R_2$ variety, with $n \geq k+3$. Then either $X$ is

1. a two-parameter family of $\mathbb{P}^{k-2}$; or
2. a one-parameter family of quadrics; or
3. a linear section of $\mathbb{G}(1, 4)$.

Notice that the hypothesis that $X$ is of type $R_2$ carries some additional hypotheses with it. First, it requires that $X$ is irreducible. In the previous classification theorems, the word irreducible was rather unimportant, since $\dim F_1(X) = 2k-2$ means that $X$ *contains* a $k$-plane rather than simply *is* a $k$-plane. Here, however, we could have a situation where $X$ is the union of a $k$-dimensional swept out by a $(k-1)$-dimensional family of lines and a $\mathbb{P}^{k-1}$. Then $X$ is swept out by its Fano variety, and $\dim F_1(X) = 2k-4$, however, such an $X$ is not at all the type of variety we are interested in classifying.

The other requirement that 'type $R_2$' carries with it is that $X$ must be swept out by its Fano variety. In theorem 1 and theorem 3, this was not important to specify because $X$ was automatically swept out by its Fano variety. Indeed, if the Fano variety swept out a proper subvariety $X' \subset X$ then $\dim X' \leq k-1$ and so the dimension of its Fano variety would violate the maximum possible. Here, however, we do not have that luxury. In particular, if $X$ is a $k$-dimensional variety containing exactly one $\mathbb{P}^{k-1}$, and no other lines, then $\dim F_1(X) = 2k-4$. Such varieties do exist, An example is the projection of the Veronese $k$-fold from its general point.

## 7.4 The Strategy

The overall approach to proving this theorem is straightforward enough. We start from the observation that if $X$ is a type $R_2$ variety that is covered by its Fano variety, then the general point $p \in X$ has a $(k-3)$-family of lines passing through it, and so $\dim X_p = k-2$ (recall $X_p \subset \mathbb{P}^n$ is the variety swept out by the lines in $X$ passing through $p$). Since $X_p \subset \mathcal{B}_p$, we get that the base locus at a general point has large dimension, which means that $\dim II_p$ is small for general $p$. We exploit this fact to show that if $X$ is not standard, then it must be that $X_p$ is irreducible for general $p$.

The next phase of the proof uses the fact that $X_p$ is irreducible for general $p$ to show that the tangent spaces to $X$ along a general line in $X$ spans either a $\mathbb{P}^k$, a $\mathbb{P}^{k+1}$ or a $\mathbb{P}^{k+2}$. The remainder of the proof consists of examining the cases individually.

## 7.5 Some Notation

Throughout the proof, we will let $X \subset \mathbb{P}^n$ be an irreducible, $k$-dimensional variety swept out by its Fano variety $\Sigma = F_1(X) \subset \mathbb{G}(1,n)$ where $\dim \Sigma = 2k-4$. For any point $p \in X$, let $\Sigma_p \subset \Sigma$ be the locus of lines in $\Sigma$ passing through $p$. To compute the dimension of $\Sigma_p$, examine the variety

$$Z = \{(p,l) : p \in l\} \subset X \times \Sigma,$$

and its projections $\pi_1 : Z \to X$ and $\pi_2 : Z \to \Sigma$. Clearly $\pi_1$ and $\pi_2$ are both surjective. Also, $\pi_1^{-1}(p) = \Sigma_p$. Therefore, for general $p \in X$, we have

$$\dim \Sigma_p = \dim \Sigma - \dim X + 1 = k - 3.$$

Let $X_p \subset \mathbb{P}^n$ be the variety swept out by $\Sigma_p$. As we have seen previously, $\dim X_p = k - 2$.

**Note.** Notice that the converse of the above statements also holds. That is, if $X \subset \mathbb{P}^n$ is an irreducible variety such that through a general point $p \in X$ there passes a $(k-3)$-dimensional variety of lines contained in $X$, then $X$ is of type $R_2$. Similarly, if for general $p \in X$, $\dim X_p = k-2$, then $X$ is of type $R_2$.

## 7.6 A Useful Theorem

As with the previous classification theorem, we begin by taking advantage of the propositions from the second fundamental form discussion. In particular, the claim we want is the following.

**Claim.** Let $X \subset \mathbb{P}^n$ be a variety of type $R_2$. Then

1. if $\dim II_p = 0$, then $X \subset \mathbb{P}^{k+1}$;
2. if $\dim II_p = 1$, $k \geq 3$ and $n > k+2$, then $X$ is standard.
3. if $\dim II_p = 2$ and $n > k+2$, then either $X$ is standard, or $X \subset \mathbb{P}^{k+3}$.

**Note.** We will only prove 3, although 1 and 2 may be proven using the same techniques (and more easily).

*Proof of 3.* We use the proposition 1. Since $\dim II_p = 2$, there are four possible values of $h$. Either $h = 0, 1, 2$ or 3.

The case when $h = 0$ is easily handled since if $h = 0$, then $X \subset \mathbb{P}^m$ and $m \leq k+3$. Therefore, $X \subset \mathbb{P}^{k+3}$.

If $h \neq 0$, then $X$ is a subvariety of a positive dimensional family of $m$-planes. We make the observation that for the general $m$-plane, $\Gamma \subset Z = \infty^h \mathbb{P}^m$, we have that $\dim(X \cap \Gamma) = k - h$. To see this, we examine the variety

$$\Phi = \{(p, \Gamma) : p \in \Gamma\} \subset X \times Y,$$

where $Y \subset \mathbb{G}(m, n)$ is the $h$-dimensional variety in the Grassmannian parametrizing $Z$. Since both projections are surjective, and the general point $p \in X$ is contained in finitely many $m$-planes of $Y$ (since otherwise, $X$ could be covered by a subfamily of $m$-planes), the theorem of the fibers gives us that for general $\Gamma \in Y$,

$$\dim(X \cap \Gamma) = \dim X - \dim Y = k - h.$$

The key idea to prove this claim when $h \neq 0$ is to examine the variety, $X_\Gamma$, swept out by the lines in $X$ which intersect a general $m$-plane $\Gamma \subset Z$. Clearly $X \cap \Gamma \subset X_\Gamma \subset X$, and so

$$k - h \leq \dim X_\Gamma \leq k.$$

Furthermore, if $l \subset X_\Gamma$ is any line contained in $X$ that intersects $\Gamma$ at a general point $p$, we have

$$l \subset \mathbb{T}_p X \subset \mathbb{T}_p Z \subset \mathbb{P}^{k-h+3},$$

where the final containment is due to the previous result. In particular, $X_\Gamma$ cannot equal $X$ because this would say that $X$ would be contained in a $\mathbb{P}^{k-2}$ (since $h \neq 0$). Therefore, we have

$$k - h \leq \dim X_\Gamma \leq k - 1.$$

Next, note that since $\Gamma \in Y$ was a general $m$-plane in the family, the general point of $X \cap \Gamma$ is a general point of $X$. Therefore, since $X$ is a type $R_2$ variety, the general point of $X_\Gamma$ has a $(k-3)$-dimensional family of lines passing through it. We now examine the cases of $h = 1, 2, 3$ separately.

If $h = 1$ then $\dim X_\Gamma = k - 1$ and so $\dim X_\Gamma$ is a type $R_1$ variety. Therefore, since the general point of $X$ is contained in a $(k-1)$-dimensional variety of type $R_1$, $X$ is a one parameter family of varieties of type $R_1$ and so $X$ is standard.

If $h = 2$ then $k - 2 \leq \dim X_\Gamma \leq k - 1$. If $\dim X_\Gamma = k - 2$, then $X_\Gamma$ is type $R_0$ and so $X$ is standard (two parameter family of varieties of type $R_0$). Similarly, if $\dim X_\Gamma = k - 1$ then $X_\Gamma$ is type $R_1$ and so $X$ is standard.

If $h = 3$, then $k - 3 \leq \dim X_\Gamma \leq k - 1$. Clearly $\dim X_\Gamma \neq k - 3$ since the general point of $X_\Gamma$ has a $(k - 3)$-dimensional family of lines passing through it. If $\dim X_\Gamma = k - 2$, then $X_\Gamma$ is type $R_0$ and so $X$ is standard. Similarly, if $\dim X_\Gamma = k - 1$ then $X_\Gamma$ is type $R_1$ and so $X$ is standard. $\square$

## 7.7 The Proof of the Classification Theorem (1)

In this section we prove that if $X \subset \mathbb{P}^n$ is a type $R_2$ variety with $n \geq k+3$ then for general $p \in X$, $X_p$ is irreducible. We accomplish this by examining the degree of $X_p$. We begin with the following.

**Claim.** If $X \subset \mathbb{P}^n$ is of type $R_2$ and is not a hypersurface, then either we have $X = \infty^2\mathbb{P}^{k-2}$, or else for general $p \in X$, $\deg X_p \leq 3$.

*Proof.* Let $X \subset \mathbb{P}^n$ be a $k$-dimensional, type $R_2$ variety that is not a hypersurface. Let $p \in X$ be a general point. We have $\dim \mathcal{B}_p \geq k-2$, since $X_p \subset \mathcal{B}_p$. Suppose that $\dim \mathcal{B}_p = k-1$. By the previous result, since $X$ is not a hypersurface, it must be that $\dim II_p \geq 1$. Therefore, if $\{Q_1, \ldots, Q_N\}$ are quadratic polynomials that span $II_p$, we must have $N \geq 2$. We see then that each $Q_i$ must be reducible, and all of the $Q_i$ must share a common factor. By proposition 2, in the second fundamental form section, this means that $X = \infty^2\mathbb{P}^{k-2}$ is standard.

On the other hand, if $\dim \mathcal{B}_p = k-2$ then every component of $X_p$ is contained in the intersection of two quadrics and so $\deg X_p \leq 4$.

Finally, if $\deg X_p = 4$ then $X_p$ is the intersection of two quadrics. Therefore, $\dim II_p = 1$, and so, again, by proposition 2, $X$ is standard. □

**Note.** The above claim gives us that if $X \subset \mathbb{P}^n$ is a $k$-dimensional, type $R_2$ variety with $n \geq k+3$ then for general $p \in X$, $X_p$ is irreducible. To see this, note that if $X_p$ were reducible then it would have a plane as a component (since $\deg X_p \leq 3$). However, this would mean that a general point of $X$ would have a $(k-2)$-plane passing through it, in which case $X$ would be a two parameter family of $(k-2)$-planes, which would make it standard.

## 7.8 Some Notation

Our next step is to verify that if $X \subset \mathbb{P}^n$ is a $k$-dimensional, type $R_2$ variety with $n \geq k+3$, then the tangent spaces to $X$ along a general line span either a $\mathbb{P}^k$, a $\mathbb{P}^{k+1}$, or a $\mathbb{P}^{k+2}$. The proof of this fact is relatively straightforward, however it requires the development of some terminology. In this section we loosen the restriction that $X$ is a type $R_2$ variety, and we study the more general situation in which a $k$-dimensional variety $X$ is covered by an irreducible $(k-1)$-dimensional family of lines.

Let $\Sigma \subset \mathbb{G}(1,n)$ be an irreducible $(k-1)$-dimensional family of lines, and let $X \subset \mathbb{P}^n$ be the variety swept out by $\Sigma$. For a general line $l \in \Sigma$, let $\Sigma_l$ be the closure in the Grassmannian $\mathbb{G}(1,n)$ of the set of lines in $\Sigma$ that intersect $l$ at a general point. Let $X_l \subset \mathbb{P}^n$ be the variety swept out by $\Sigma_l$.

**Note.** For general $l \in \Sigma$ we have that

$$\Sigma_l = \overline{\bigcup_{p \in l \setminus S} \Sigma_p},$$

where $S \subset l$ is a fixed finite subset (recall $\Sigma_p$ is the set of lines in $\Sigma$ passing through $p$). Therefore, $\Sigma_l$ has no isolated points and so it is equidimensional with each component being one-dimensional.

Next, let $\omega_l \subset \mathbb{P}^n$ be the plane spanned by the family of tangent planes to $X$ at smooth points of $l$.

**Note.** Clearly for general $l \in \Sigma$ we have $X_l \subset \omega_l$.

Now, since every point of $X$ has a line in $\Sigma$ passing through it, for a general point $p \in X$ we can parametrize $X$ in a neighborhood of $p$ by

$$v(t_1, \ldots, t_k) = v(T) = [v_0(T), \ldots, v_n(T)],$$

where $v(0) = p$ and

$$v_i(t_1, \ldots, t_k) = x_i(t_1, \ldots, t_{k-1}) + t_k \cdot y_i(t_1, \ldots, t_{k-1}) = x_i(T') + t \cdot y_i(T'),$$

where $y(T')$ is a point (other than $x(T')$) on the line in $\Sigma$ through $x(T')$. Therefore, if $l \in \Sigma_p$ is a general line through $p$ (note $l \in \Sigma$ is a general line since $p \in X$ is a general point), the tangent space $\mathbb{T}_p X$ is the $k$-plane spanned by the points

$$\frac{\partial x}{\partial t_i}(0) + t\frac{\partial y}{\partial t_i}(0), \text{ for } i = 1, \ldots, k-1; \text{ and } l.$$

In this way we may define rational maps

$$\begin{aligned} \pi_i : \quad & l \dashrightarrow \mathbb{P}^n \\ & q = x(0) + ty(0) \mapsto \frac{\partial x}{\partial t_i}(0) + t\frac{\partial y}{\partial t_i}(0), \end{aligned}$$

for $i = 1, \ldots, k-1$, which we collect into a single rational map

$$\begin{aligned} \phi : \quad & l \dashrightarrow \mathbb{G}(k-2, n) \\ & q \mapsto \operatorname{Span}\{\pi_1(q), \ldots, \pi_{k-1}(q)\}. \end{aligned}$$

Let $\Theta(l) \subset \mathbb{G}(k-2, n)$ be the closure in $\mathbb{G}(k-2, n)$ of the image of $\phi$. Let $X_{\Theta(l)} \subset \mathbb{P}^n$ be the variety swept out by $\Theta(l)$.

**Note.** The linear space $\omega_l$ is exactly the linear space spanned by $X_{\Theta(l)}$ and $l$.

## 7.9 Proof of the Classification Theorem (2)

In this section, we prove that if $X \subset \mathbb{P}^n$ is a $k$-dimensional, type $R_2$ variety with $n \geq k+3$, then the family of tangent planes to $X$ along a general line in $X$ spans either a $\mathbb{P}^k$, a $\mathbb{P}^{k+1}$ or a $\mathbb{P}^{k+2}$. In light of the above notation, we must show that for general $l \in \Sigma$, we have

$$k \leq \dim \omega_l \leq k+2.$$

We start with the following observation.

**Note.** Each $\pi_i(l)$ is either a point or a dense open subset of a line. Therefore, since $X_{\Theta(l)}$ is contained in the linear space spanned by the $\pi_i(l)$, we have that $X_{\Theta(l)}$ is contained in, at most, a $\mathbb{P}^{2k-3}$, and so $\omega_l$ is contained in, at most, a $\mathbb{P}^{2k-1}$.

**Claim.** If every $\mathbb{P}^{k-2}$ in $\Theta(l)$ contains a fixed $\mathbb{P}^m$, then $X_{\Theta(l)}$ is contained in at most a $\mathbb{P}^{2k-m-2}$, and so $\omega_l$ is contained in at most a $\mathbb{P}^{2k-m}$.

*Proof.* If every $\mathbb{P}^{k-2}$ of $\Theta(l)$ contains a fixed $\mathbb{P}^m$ then this $\mathbb{P}^m$ along with $\pi_i(l)$ for $k-m-2$ suitably chosen $i$ will span $X_{\Theta(l)}$ (any $i$ such that $\pi_i(l)$ is not contained in the $\mathbb{P}^m$ will do; note that there must be at least $k-m-2$ such $i$). Therefore, $X_{\Theta(l)}$ is contained in a $\mathbb{P}^{2k-m-2}$, as desired. □

**Claim.** Let $X \subset \mathbb{P}^n$ be a $k$-dimensional variety covered by an $r$-dimensional family of lines, $\Sigma \subset \mathbb{G}(1,n)$. Then for the general $l \in \Sigma$, we have that $\dim \omega_l \leq 2k-m$, for some $m$ such that $r-k+2 \leq m \leq k$.

*Proof.* Since $X$ is $k$-dimensional and covered by an $r$-dimensional family of lines, we must have $r \geq k-1$. In particular, there must be a $(k-1)$-dimensional subvariety of $\Sigma$ that sweeps out $X$ (the intersection of $\Sigma$ with a general choice of hyperplanes $H_1, \ldots, H_{r-k+1} \subset \mathbb{P}^N$ will yield such a family). By the previous claim, it suffices to show that for the general $l \in \Sigma$, the tangent $k$-planes to $X$ along the smooth points of $l$ have a fixed $\mathbb{P}^m$ in common for some $m$ such that $r-k+2 \leq m \leq k$.

So choose such a general $l \in \Sigma$ and let $p \in l$ be a general point. Clearly $X_p \subset X$ and so $\mathbb{T}_q X_p \subset \mathbb{T}_q X$ for any $q \in l$. Therefore, the plane spanned by the family of tangent spaces to $X$ along $l$ contains the plane spanned by the family of tangent spaces to $X_p$ along $l$. However, $X_p$ is a cone with vertex $p$, and so the family of tangent spaces along a general line through $p$ (such as $l$) will be constant. Therefore $\mathbb{T}_p X_p \subset \mathbb{T}_q X$ for every $q \in l$. Finally, since $\dim X_p = r-k+2$ and $\mathbb{T}_p X_p \subset \mathbb{T}_p X$ we have

$$r-k+2 \leq \dim \mathbb{T}_p X_p \leq k,$$

and so $\mathbb{T}_p X_p$ is our desired $m$-plane. □

The following follows immediately from the previous result. We state it as a corollary because we will refer to it frequently.

**Corollary 1.** When $X \subset \mathbb{P}^n$ is a $k$-dimensional, type $R_2$ variety, with $n \geq k+3$, the above gives us exactly what we wanted. Namely, we see that the family of tangent spaces to $X$ along a general line $l$ spans either

1. a $\mathbb{P}^k$; or
2. a $\mathbb{P}^{k+1}$; or
3. a $\mathbb{P}^{k+2}$.

The above result should not be too surprising. Since our standard varieties of type $R_2$ contain planes of high dimension, and so as you move along a line, you are likely staying within a given plane of high dimension, which means that a large portion of the tangent space does not change.

Theorem 4, therefore, follows from the following three theorems.

**Theorem 5.** Let $X \subset \mathbb{P}^n$ be a $k$-dimensional, type $R_2$ variety, with $n \geq k+3$ such that for general $l \in \Sigma$, $\dim \omega_l = k$. Then $X = \infty^2\mathbb{P}^{k-2}$.

**Theorem 6.** Let $X \subset \mathbb{P}^n$ be a $k$-dimensional, type $R_2$ variety, with $n \geq k+3$ such that for general $l \in \Sigma$, $\dim \omega_l = k+1$. Then $X$ is standard.

**Theorem 7.** Let $X \subset \mathbb{P}^n$ be a $k$-dimensional, type $R_2$ variety, with $n \geq k+3$ such that for general $l \in \Sigma$, $\dim \omega_l = k+2$. Then either $X$ is standard or $X$ is a linear section of $\mathbb{G}(1,4)$.

Before embarking on the proofs of any of the three theorems, we break to verify that for general $l \subset \mathbb{G}(1,4)$, $\dim \omega_l = 8$.

## 7.10 Another Look at $\mathbb{G}(1,4)$

Notice that $\mathbb{G}(1,4)$ does fall into the third case of corollary 1. To see this, we choose a point $\Lambda \in \mathbb{G}$ and we note that

$$\mathbb{T}_\Lambda \mathbb{G} = \{\eta : \eta \wedge \lambda = 0\},$$

where $\lambda$ is the two-form corresponding to $\Lambda$. Therefore, for some line $L \subset \mathbb{G}$, we have

$$X_L = \bigcup_{\Lambda \in L} \mathbb{T}_\Lambda \mathbb{G} = \{\eta : \eta \wedge \left(\wedge^3 \Gamma\right) = 0\},$$

where $\Gamma \subset \mathbb{P}^4$ is the 2-plane swept out by the lines in $L$. Clearly $X_L$ is linear as it is the kernal of a linear map. Also, $\dim X_L = 8$ since if we take a basis $\{v_1, v_2, v_3\}$ for $\widetilde{\Gamma}$ and extend it to a basis $\{v_1, v_2, v_3, v_4, v_5\}$ for $\mathbb{A}^5$, then the only basis element of $\wedge^2\mathbb{A}^5$ that is not in $X_L$ is $x_4 \wedge x_5$. Therefore, $\dim X_L = 8$.

## 7.11 Proof of theorem 5

In this section we prove theorem 5. We restate the theorem.

**Theorem.** Let $X \subset \mathbb{P}^n$ be a $k$-dimensional, type $R_2$ variety, with $n \geq k+3$ such that for general $l \in \Sigma$, $\dim \omega_l = k$. Then $X = \infty^2\mathbb{P}^{k-2}$.

Before beginning the proof, we briefly revisit the Gauss map to cite a result we will need.

### 7.11.1 The Gauss Map Revisited

We will need the following result concerning the fibers of the Gauss map.

**Proposition 3.** The Gauss map of $X$ has $m$-dimensional fibers if and only if, for the general $p \in X$, all of the quadrics of $II_p$ are singular along a fixed $\mathbb{P}^{m-1}$. Moreover, such a $\mathbb{P}^{m-1}$ in $\mathbb{T}_p X$ represents a $\mathbb{P}^m$ on $X$ that is the fiber of the Gauss map.

A proof is contained in [G-H].

As a corollary, we have the following neat result. A direct proof of which can also be found in [Z].

**Fact.** If $X \subset \mathbb{P}^n$ is smooth, then the fibers of the Gauss map are finite.

### 7.11.2 Proving Theorem 5

*Proof of Theorem 5.* Let $l \in \Sigma$ be a general line and let $\Sigma_l \subset \Sigma$ be the set of lines which intersect $l$. Then $\dim \Sigma_l = k-2$ since

$$\Sigma_l = \bigcup_{p \in l} \Sigma_p.$$

Let $X_l \subset \mathbb{P}^n$ be the variety swept out by $\Sigma_l$. We have $k-2 \leq \dim X_l \leq k-1$. Since the tangent space to $X$ is fixed along general lines, the tangent space to $X$ is fixed along all of $X_l$. Therefore, $X_l$ is contained in the fiber of the Gauss map, which must be linear by proposition 3. Therefore, the fibers of the Gauss map are either $\mathbb{P}^{k-2}$ or $\mathbb{P}^{k-1}$. In the first case, $X = \infty^2\mathbb{P}^{k-2}$, and in the second case $X$ is of type $R_1$, contrary to our hypotheses. $\square$

### 7.11.3 An Aside

It seems that in every subject of mathmatics there are certain facts that every student who a class in the subject will get asked to prove at one point or another. Examples of such facts in algebra and analysis are

1. If $R$ is a finite integral domain prove that $R$ is a field.
2. Prove that every finite field has order $p^m$ for some prime $p$.
3. If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic such that $f(z) \in \mathbb{R}$ for every $z \in \mathbb{C}$, prove that $f$ is constant.

In algebraic geometry, an example of such a fact is the following.

**Fact.** If $X \subset \mathbb{P}^n$ is a hypersurface other than a plane and $\Lambda \subset X$ is an $r$-plane with $r \geq \frac{n}{2}$, then $X$ must be singular.

We offer a proof of this fact using the Gauss map. We will use the observation above that if $X$ is smooth, then the Gauss map has finite fibers. We will also use the fact that if $\varphi : \mathbb{P}^n \to \mathbb{P}^m$ is a regular map and $m < n$, then $\varphi$ is constant.

*Proof.* Let $X \subset \mathbb{P}^n$ be a smooth hypersurface other than a plane cut out by the polynomial $F$. Let $\Lambda \subset X$ be an $r$-plane. Then for any $p \in \Lambda$ we have $\Lambda \subset \mathbb{T}_p X$.

If $X \subset \mathbb{P}^n$ is a smooth hypersurface then the Gauss map, $\mathcal{G} : X \to \mathbb{G}(n-1, n)$ is regular. Let $\mathbb{G}_\Lambda$ be the set of hyperplanes in $\mathbb{P}^n$ that contain $\Lambda$. Then the Gauss map restricts to a map

$$\mathcal{G}\big|_\Lambda : \Lambda \to \mathbb{G}_\Lambda$$

(notice that $\mathcal{G}\big|_\Lambda$ is regular since $X$ is smooth). Since $\mathbb{G}_\Lambda$ may be seen as the set of linear forms on $\mathbb{P}^n$ which vanish on $\Lambda$, we see that $\mathbb{G}_\Lambda \simeq \mathbb{P}^{n-r}$. Therefore, we have a regular map

$$\mathcal{G}\big|_\Lambda : \mathbb{P}^r = \Lambda \to \mathbb{G}_\Lambda = \mathbb{P}^{n-r}.$$

If $r \geq \frac{n}{2}$ then we would have $r \geq n - r$ and so $\mathcal{G}\big|_\Lambda$ would have to be constant, contradicting the fact that when $X$ is smooth, the fibers of the Gauss map are finite. $\square$

## 7.12 Proof of Theorem 6

We now prove theorem 6, rewritten below.

**Theorem.** Let $X \subset \mathbb{P}^n$ be a $k$-dimensional, type $R_2$ variety, with $n \geq k+3$ such that for general $l \in \Sigma$, $\dim \omega_l = k+1$. Then $X$ is standard.

### 7.12.1 Our Approach

The second case of corollary 1 is more difficult to handle than the first case, and our proof is more complicated. Our goal is to show that if $X \subset \mathbb{P}^n$ is a $k$-dimensional, type $R_2$ variety with $n \geq k+3$, and furthermore is such that for general $l \in \Sigma$, $\dim \omega_l = k+1$, then $X$ is standard. We begin by noting that $X_p \subset \omega_l$, and so for general $l \in \Sigma$, $k-2 \leq \dim(\omega_l \cap X) \leq k-1$. Our first move will be to prove that if for general $l \in \Sigma$, $\dim(\omega_l \cap X) = k-2$ then $X = \infty^2 \mathbb{P}^{k-2}$. We then are able to proceed with the assumption that for general $l \in \Sigma$, $\dim(\omega_l \cap X) = k-1$. This makes $X$ a $k$-dimensional variety containing a family of $(k-1)$-dimensional varieties (namely the $\omega_l \cap X$), each of which is contained in a $\mathbb{P}^{k+1}$ (namely $\omega_l$). Moreover, the dimension of this family is at least $k-1$, since a $(k-1)$-dimensional family of lines is needed to cover $X$. We will use help from another classification to help us classify $X$ based on this observation. We introduce these theorems presently.

### 7.12.2 Del Pezzo's Helpsul Result

**Definition.** A *Steiner surface* is the image of the Veronese surface in $\mathbb{P}^5$ under projection to $\mathbb{P}^3$ from a disjoint 2-plane in $\mathbb{P}^5$.

**Definition.** We say that a variety $X$ is an *extension* of $Y$ if there exists a set of hyperplanes $H_1, \ldots, H_m$ such that $Y = X \cap H_1 \cap \cdots \cap H_m$. If $Y$ has no extensions other than a cone, we say that $Y$ is not *extendable.*

We now cite two facts, the first of which is due to Castelnuovo and Kronecker. For proofs of these facts see [R2].

**Fact.** Let $X \subset \mathbb{P}^3$ be a surface containing a 2-dimensional family of hyperplane sections which are reducible or nonreduced. Then either $X$ is ruled or $X$ is a Steiner surface.

**Fact.** The Veronese surface in $\mathbb{P}^5$, the Veronese surface in $\mathbb{P}^4$ and the Steiner surface in $\mathbb{P}^3$ are not extendible.

**Definition.** A *surface section* of a $k$-dimensional variety $X \subset \mathbb{P}^n$ is a section of $X$ with an $(n-k+2)$-plane.

**Claim.** Let $X \subset \mathbb{P}^n$ be a $k$-dimensional variety other than a $k$-plane such that its general surface section is ruled. Then $\dim F_1(X) = 2k-3$.

*Proof.* Let $\mathbb{G} = \mathbb{G}(n-k+2, n)$ be the Grassmannian of $(n-k+2)$-planes in $\mathbb{P}^n$. Define the variety

$$Z = \{(l, \Gamma) : l \subset \Gamma \cap X\} \subset F_1(X) \times \mathbb{G}$$

with projections $\pi_1 : Z \to F_1(X)$ and $\pi_2 : Z \to \mathbb{G}$. Clearly $\pi_1$ is surjective, and we are told that $\pi_2$ maps onto a dense subset of $\mathbb{G}$. Therefore, for general $l \in F_1(X)$ and $\Gamma \in \mathbb{G}$, we have

$$\dim F_1(X) + \dim \mathbb{G}_l = \dim Z = \dim \mathbb{G} + \dim \pi_2^{-1}(\Gamma),$$

and so $\dim F_1(X) = 2k - 4 + \dim \pi_2^{-1}(\Gamma)$. Finally, $\pi_2^{-1}(\Gamma) \simeq F_1(X \cap \Gamma)$, and so since the general surface section of $X$ is ruled, $\dim \pi_2^{-1}(\Gamma) = 1$. The result follows. $\square$

**Lemma 1.** Let $n > 3$ and let $X \subset \mathbb{P}^n$ be a $k$-dimensional variety. If there exists an $(n-k+1)$-dimensional family of reducible or non reduced hyperplane sections, then $X = \infty^1 \mathbb{P}^{k-1}$ or it is a cone over a Steiner surface in $\mathbb{P}^3$ or a cone over the Veronese surface in $\mathbb{P}^4$ or $\mathbb{P}^5$.

*Proof.* Let $X \subset \mathbb{P}^n$ be a $k$-dimensional variety with an $(n-k+1)$-dimensional family of reducible or non reduced hyperplane sections. By generically projecting $X$ onto a hypersurface in $\mathbb{P}^{k+1}$. The image will have the property that the general surface section will be a surface in $\mathbb{P}3$ with a two-dimensional family of reducible or non reduced hyperplane sections. Therefore, by the fact by Castelnuovo and Kronecker, the general surface section is either ruled or a Steiner surface. By the previous claim, the image of the projection either a cone over a Steiner surface or a scroll in $\mathbb{P}^{k-1}$. Therefore our original $X$ was either a cone over a Steiner surface, or a cone over a Veronese surface or else it was $\infty^1 \mathbb{P}^{k-1}$, as desired. $\square$

**Proposition 4** (Del Pezzo)**.** Let $X \subset \mathbb{P}^n$ be a $k$-dimensional variety such that its section with its general tangent hyperplane is $(k-1)$-dimensional. Then $X \subset \mathbb{P}^3$, or $X$ is ruled.

*Proof.* We examine the case when $k = 2$ and $k > 2$ separately.

**Step 1:** $k = 2$

Let $X \subset \mathbb{P}^n$ be a surface such that the section with the general tangent hyperplane is a curve. If $X$ were not in $\mathbb{P}^3$ then its sections with tangent hyperplanes at general points would all be reducible. The family of tangent hyperplane sections is therefore an $(n-1)$-dimensional family of reducible hyperplane sections, and so by lemma 1, $X$ is either a Veronese surface or it is ruled (since we assumed that $X \not\subset \mathbb{P}^3$, and so $X$ can't be a Steiner surface). We now check that $X$ can't be a Veronese surface.

Since the Veronese surface in $\mathbb{P}^5$ and in $\mathbb{P}^4$ are both of degree 4, the general hyperplane section of either will be a quartic curve. Therefore, if the general tangent hyperplane section were a curve, it would split into two irreducible conics (since Veronese surfaces do not contain lines). and therefore the general tangent line would intersect the general hyperplane section in another point. This can't happen for the Veronese surface in $\mathbb{P}^5$ because the general hyperplane section is a rational normal curve of degree 4, and so it has no tritangent lines. This can't happen for the Veronese surface in $\mathbb{P}^4$ because the general hyperplane section is contained in a quadric (since the general hyperplane section is a rational quartic in $\mathbb{P}^3$), and so all the tangent lines would be rulings of such a quadric. However, the rulings of a quadric do not envelope a curve.

**Step 2:** $k > 2$

If $X$ is $k$-dimensional with $k > 2$, then the general surface section of $X$ will be so that its intersection with its general tangent plane is a curve. Therefore, by the above, either the surface section will be in $\mathbb{P}^3$, or it is ruled. Therefore, either $X \subset \mathbb{P}^{k+1}$ or $X$ is a scroll in $\mathbb{P}^{k-1}$. □

We use Del Pezzo's result to prove the classification result that we will use to help prove theorem 6.

### 7.12.3 Some Helpful Classification Results

**Proposition 5.** Let $X \subset \mathbb{P}^n$ be a surface containing an $(n-1)$-dimensional family of irreducible curves, each of which spans a $\mathbb{P}^{n-2}$. Then $X$ is contained in $\mathbb{P}^{n-1}$.

*Proof.* Let $X \subset \mathbb{P}^n$ be a surface containing an $(n-1)$-dimensional family of irreducible curves, $\mathcal{F}$, where each curve in $\mathcal{F}$ is contained in a $\mathbb{P}^{n-2}$. By using the theorem of the fibers on the projection maps of the variety

$$Z = \{(p, C) : p \in C\} \subset X \times \mathcal{F},$$

we see that $\dim \mathcal{F}_p = n-2$ where $\mathcal{F}_p \subset \mathcal{F}$ is the set of curves passing through $p$. If we let $\pi_p : X \dashrightarrow \mathbb{P}^{n-1}$ be the projection through $p$, then $\pi_p(X) \subset \mathbb{P}^{n-1}$ is a surface containing an $(n-2)$-dimensional family of irreducible curves, each of which is contained in a $\mathbb{P}^{n-3}$. If we project from $(n-3)$ general points, therefore, we wind up with a surface in $\mathbb{P}^3$ containing a 2-dimensional family of plane curves. Therefore, this surface in $\mathbb{P}^3$ is a plane, and so $X$ is contained in the linear space spanned by this plane and the $(n-3)$ points of projection. □

**Claim.** Let $X \subset \mathbb{P}^n$ be a ruled surface other than a plane. Let $x \in X$ be a general point, and let

$$\pi_x : X \dashrightarrow \mathbb{P}^{n-1}$$

be projection from $x$. Let $Y = \pi_x(X) \subset \mathbb{P}^{n-1}$. Then if $X$ is not a cone and $Y$ is not a plane, $\pi_x$ is birational.

*Proof.* Suppose $\pi_x$ is not birational. Then for a general point $p \in X$, there exists a $p' \in X$ such that $\pi_x(p) = \pi_x(p')$. Let $S = \{l_1, \dots, l_N\}$ be the set of lines in $X$ through $p$, and, similarly, let $S' = \{l'_1, \dots, l'_M\}$ be the set of lines in $X$ through $p'$. Since $\deg Y = \deg X - 1$, there must exist $i$ and $j$ such that $\pi_x(l_i) = \pi_x(l'_j)$. Let $l = l_i$, $l' = l'_j$, and let $q = \pi_x(p)$. Also let $l_* = \pi_x(l)$.

As we have set it up, both $l$ and $l'$ are contained in the plane spanned by $q$ and by $l_*$. Therefore $l$ and $l'$ intersect each other.

For fixed $p \in X$ we can define the variety

$$Z_p = \{(x, p') : \pi_x(p) = \pi_x(p')\} \subset X \times X.$$

Clearly projection onto the first coordinate is surjective and finite. Therefore, if the projection onto the second coordinate were not surjective, then it would have one-dimensional fibers, which would mean that the line spanned by $p$ and $p'$ were contained in $X$. This is a contradiction because $p'$ is general and $X$ is not a plane.

We see, therefore, that the lines of $X$ intersect each other. Since they will intersect in a fixed point, $X$ is a cone, contradicting our hypotheses. Therefore, when $X$ is not a cone, $\pi_p$ is birational. □

**Proposition 6.** Let $X \subset \mathbb{P}^n$ be a surface containing a $k$-dimensional family of curves in $\mathbb{P}^k$. Then either

1. $X \subset \mathbb{P}^{k+1}$; or
2. $X$ is a cone; or
3. $k = 2$ and $X$ is a Veronese surface in $\mathbb{P}^4$ or $\mathbb{P}^5$; or
4. $X$ is a rational normal scroll in $\mathbb{P}^n$.

*Proof.* Suppose $X$ is not contained in a $\mathbb{P}^{k+1}$. Then we can birationally project $X$ to a $\mathbb{P}^{k+2}$. Since $X$ contains a $k$-dimensional family of curves in $\mathbb{P}^k$, it has a $(k+1)$-dimensional family of reducible hyperplane sections. Therefore, by lemma 1 from the section on Del Pezzo's theorem, $X$ is either a cone, or a Veronese surface, or a ruled surface. If the general surface section of $X$ ruled and is not a cone, then by the previous claim we can project $X$ from $(k-1)$ general points of $X$, birationally to a surface $X' \subset \mathbb{P}^3$. The curves in $X$ passing through the points of projection (there is a one-dimensional family passing through each) are mapped to lines in $X'$. Since the projection is birational, we see that $X'$ is ruled in two different ways, and so it is a quadric hypersurface. Therefore, $X$ is a rational normal scroll in $\mathbb{P}^n$. Moreover, the $k$-dimensional family of curves in $X$ are the rational normal curves of degree $k$ on the rational normal scroll. $\square$

**Note.** We developed Del Pezzo's theorem from scratch and use it to prove proposition 6. Emilia Mezzetti takes the opposite approach in [M]. He uses the language of schemes to prove proposition 6, and then deduces Del Pezzo's theorem.

### 7.12.4 Proof of Theorem 6

We now prove theorem 6, rewritten again below.

**Theorem.** Let $X \subset \mathbb{P}^n$ be a $k$-dimensional, type $R_2$ variety, with $n \geq k+3$ such that for general $l \in \Sigma$, $\dim \omega_l = k+1$. Then $X$ is standard.

*Proof of Theorem 6.* We divide the proof into several steps.

**Step 1: Computing** $\dim(\omega_l \cap X)$

We have already noted that for any line $l \in \Sigma$ and a point $p \in l$, we have

$$X_p \subset \omega_l \cap X \subset X,$$

and so for general $l$,

$$k-2 \leq \dim(\omega_l \cap X) \leq k-1,$$

since it can't be that $X \subset \omega_l$ since $n \geq k+3$.

Recall that for $p \in l$,

$$X_p \subset X_l \subset \omega_l \cap X.$$

Therefore, if for general $l \in \Sigma$, $\dim(\omega_l \cap X) = k-2$, then for general $p \in l$, $\dim X_l = k-2$. Since $X_p$ is irreducible, it must be a component of $X_l$. This tells us that for general $p, q \in l$, we have $X_p = X_q$, and so $X_p$ is a cone with $p$ and $q$ as vertices. Therefore, $X_p$ is a $(k-2)$-plane, and $X$ is standard.

**Step 2: A Bit of Notation**

Whether or not this section should actually count as a step in the proof is debatable. We choose to include it as a step both because the notation it introduces will be crucial for the remainder of the proof, and because we make several important observations.

Let $\Omega$ be the closure in the suitable Grassmannian of the family of $\omega_l$ for general $l \in \Sigma$. Define the map

$$\psi : \Sigma \dashrightarrow \Omega : l \mapsto \omega_l.$$

For any $\omega \in \Omega$, let $\Sigma_\omega = \psi^{-1}(\omega)$. That is,

$$\Sigma_\omega = \{l \in \Sigma : \omega_l = \omega\}.$$

**Note.** By using the theorem of the fibers on $\psi$ we get that

$$\dim \Sigma_\omega = \dim \Sigma - \dim \Omega.$$

Let $X_\omega \subset \mathbb{P}^n$ be the variety swept out by $\Sigma_\omega$. Also define $\Phi(\omega) \subset \Sigma$ to be the set of lines which intersect $X_\omega$. Let $X_{\Phi(\omega)} \subset \mathbb{P}^n$ be the variety swept out by $\Phi(\omega)$.

**Note.** Observe that

$$\Phi(\omega) = \bigcup_{p \in X_\omega} \Sigma_p,$$

and so $X_{\Phi(\omega)} \subset \omega$ since $\mathbb{T}_p X \subset \omega$ for every $p \in X_\omega$.

**Note.** Also worth noting is the fact that $X_\omega \subset X_{\Phi(\omega)}$.

**Step 3: A Lower Bound for $\dim \Omega$**

**Claim.** Let $X \subset \mathbb{P}^n$ be a $k$-dimensional, type $R_2$ variety with $n \geq k+3$. Then either $X$ is standard or $\dim \Omega \geq 4$.

*Proof.* Suppose $\dim \Omega \leq 3$. Then $\dim \Sigma_\omega = \dim \Sigma - \dim \Omega \geq 2k-7$. Therefore, $\dim X_\omega \geq k-2$, since otherwise $X_\omega$ would have a Fano variety of too high a dimension. This tells us that $k-2 \leq \dim X_{\Phi(\omega)} \leq k$.

If $\dim X_{\Phi(\omega)} = k$ then $X = X_{\Phi(\omega)}$ and so $X \subset \omega$ which can't be the case because $n \geq k+3$.

If $\dim X_{\Phi(\omega)} = k-1$ then by examining the variety

$$Z = \{(p, l) : l \in \Sigma_p\} \subset X_\omega \times \Phi(\omega),$$

and its projections, we see that $\dim \Phi(\omega) = 2k-5$. This means that $X_{\Phi(\omega)}$ is a variety of type $R_1$, and so $X$ is standard.

If $\dim X_{\Phi(\omega)} = k-2$ then $X_{\Phi(\omega)} = X_\omega$, and so through the general point of $X_\omega$, there passes a $(k-3)$-dimensional family of lines (namely all of $\Sigma_p$), and so $X_\omega$ is a $\mathbb{P}^{k-2}$, which means that $X$ is standard. $\square$

**Step 4: The Key Construction**

With all of the notation defined in step 2 come several natural rational maps. Our strategy for ultimately proving theorem 6 is to use the theorem of the fibers on two such maps. In this section, we construct the necessary rational maps and make some necessary observations which will allow us to prove the theorem in the step 5.

We refocus on the case when $X \subset \mathbb{P}^n$ is a $k$-dimensional, type $R_2$ variety with $n \geq k+3$, and where the general line $l \in \Sigma$ is such that $\dim \omega_l = k+1$ and $\dim(\omega_l \cap X) = k-1$.

For any $(n-k+2)$-plane, $\Gamma \subset \mathbb{P}^n$, define the map

$$\begin{aligned} \psi_\Gamma : \quad & \Omega \dashrightarrow \mathbb{G}(3, n-k+2) \\ & \omega \mapsto \omega \cap \Gamma. \end{aligned}$$

Clearly $\psi_\Gamma$ is regular on the open subset of $\Omega$ consisting of the $(k+1)$-planes which do not contain $\Gamma$. Let $\Omega_\Gamma = \psi_\Gamma(\Omega) \subset \mathbb{G}(3, n-k+2)$. The theorem of the fibers gives

$$\dim \Omega - \dim \Omega_\Gamma = \psi_\Gamma^{-1}(\Lambda),$$

for general $\Lambda \in \Omega_\Gamma$. We now prove a claim which allows us to compute the dimension of the general fiber of $\psi_\Gamma$ if a certain condition holds.

**Claim.** Let $\Gamma \subset \mathbb{P}^n$ be a general $(n-m)$-plane, and define

$$\Omega_\Gamma = \{\omega \cap \Gamma : \omega \in \Omega \text{ and } \omega \not\subset \Gamma\}.$$

If $\dim \Omega \leq k+2-m$, then $\dim \Omega_\Gamma = \dim \Omega$.

*Proof.* We prove this by induction on $m$. For the base case let $m = 1$. We must show that if $\dim \Omega \leq k+1$, then for a general hyperplane $H \subset \mathbb{P}^n$, we have $\dim \Omega_H = \dim \Omega$.

For a general hyperplane $H \subset \mathbb{P}^n$, define the map

$$\begin{aligned} \psi_H : \quad & \Omega \dashrightarrow \Omega_H \\ & \omega \mapsto \omega \cap H. \end{aligned}$$

The theorem of the fibers gives us that for general $\Lambda \in \Omega_H$,

$$\dim \Omega - \dim \Omega_H = \psi_H^{-1}(\Lambda).$$

Therefore, if $\dim \Omega > \dim \Omega_H$ then it must be that $\dim \psi_H^{-1}(\Lambda) \geq 1$.

Now, for $\omega \in \Omega$, define $\mathbb{G}^\omega \subset \mathbb{G}(k, n)$ to be the set of hyperplanes contained in $\omega$. Clearly, $\mathbb{G}^\omega \simeq \mathbb{G}(k, k+1)$. We examine the variety

$$Z_\omega = \{(\Lambda, \omega') : \Lambda \subset \omega'\} \subset \mathbb{G}^\omega \times \Omega$$

and its projections $\pi_1$ and $\pi_2$. Clearly $\pi_1$ is surjective. Also, note that the general fiber of $\pi_1$ is the same as the general fiber of $\psi_H$ for some hyperplane $H \subset \mathbb{P}^n$. Therefore, if we assume that $\dim \Omega > \dim \Omega_H$ then the fiber of $\pi_1$ is at least one-dimensional. This gives us $\dim Z_\omega \geq k+2$. Finally, note that the general fibers of $\pi_2$ are zero dimensional since if $\omega'$ contains two distinct hyperplanes $\Lambda, \Lambda' \subset \omega$, then $\omega'$ would have to contain their span, namely $\omega$. Therefore, we see that if $\dim \Omega > \dim \Omega_H$ then

$$k+2 \leq \dim Z_\omega = \dim \pi_2(Z) \leq \dim \Omega,$$

and so the base case is proved.

The general case is proven almost identically to the base case. Let $\Gamma \subset \mathbb{P}^n$ be a general $(n-m)$-plane. Let $H_1, \dots, H_m \subset \mathbb{P}^n$ be the general hyperplanes such that $\Gamma = H_1 \cap \cdots \cap H_m$. Define $\Gamma_i = H_1 \cap \cdots \cap H_i$, let $\Omega_i = \Omega_{\Gamma_i}$, and let $\Omega_0 = \Omega$. Note that $\Gamma_m = \Gamma$ and so $\Omega_m = \Omega_\Gamma$. We have

$$\dim \Omega_\Gamma = \dim \Omega_m \leq \cdots \leq \dim \Omega_1 \leq \dim \Omega.$$

We see that $\dim \Omega_\Gamma \neq \dim \Omega$ if and only if there exists some $i$, with $1 \leq i \leq m$ such that $\dim \Omega_i < \dim \Omega_{i-1}$. Just as we did in the base case, we define rational maps

$$\begin{aligned} \varphi_i : \quad & \Omega_{i-1} \dashrightarrow \Omega_i \\ & \lambda_{i-1} \mapsto \lambda_i = \lambda_{i-1} \cap H_i, \end{aligned}$$

where $\lambda_{i-1} = \omega \cap H_1 \cap \cdots \cap H_{i-1}$. Just as before, we see that if $\dim \Omega_{i-1} > \dim \Omega_i$ then the general fiber of $\varphi_i$ is at least one-dimensional. We then define $\mathbb{G}^{\lambda_{i-1}}$ to be the set of hyperplanes in $\lambda_{i-1}$ and we examine the variety

$$Z_{\lambda_{i-1}} = \{(\Lambda, \lambda') : \lambda' \in \Lambda\} \subset \mathbb{G}^{\lambda_{i-1}} \times \Omega_{i-1}.$$

Exactly as in the base case, we see that projection onto the first component is surjective and may be assumed to have fibers with dimension at least 1. Additionally, the projection onto the second component is finite. This tells us that if $\dim \Omega_{i-1} > \dim \Omega_i$, then $\dim \Omega_{i-1} \geq k+3-i$. Therefore, if $\dim \Omega \leq k+2-m$ then $\dim \Omega_{i-1} \leq k+2-i$ for every $i$ and so $\dim \Omega_\Gamma = \dim \Omega$, as desired. $\square$

**Note.** This claim tells us that when $\Gamma \subset \mathbb{P}^n$ is a general $(n-k+2)$-plane and $\dim \Omega \leq 4$, then $\psi_\Gamma$ is finite onto its image.

**Note.** Notice that if $\Gamma \subset \mathbb{P}^n$ is a general $(n-k+2)$-plane and $\dim \Omega_i \leq 4$ for any $i$, then

$$\dim \Omega_i = \dim \Omega_{i+1} = \cdots = \dim \Omega_\Gamma.$$

This observation combined with the bound on $\dim \Omega$ obtained in step 2, tell us that if $X \subset \mathbb{P}^n$ is a nonstandard, type $R_2$ variety then for a general $(n-k+2)$-plane, $\Gamma \subset \mathbb{P}^n$, we have that $\dim \Omega_\Gamma \geq 4$.

In order to define the other rational map of interest, note that for a general $(n-k+2)$-plane, $\Gamma \subset \mathbb{P}^n$, $\omega \cap \Gamma \cap X$ is a curve in $\omega \cap \Gamma = \mathbb{P}^3$. Let $\mathcal{F}_\Gamma$ be the family of such curves as $\omega \in \Omega$ varies. This gives us a surjective map

$$\begin{aligned} \phi_\Gamma : \quad & \Omega_\Gamma \to \mathcal{F}_\Gamma \\ & \omega \cap \Gamma \mapsto \omega \cap \Gamma \cap X. \end{aligned}$$

**Note.** If the general curve of $\mathcal{F}_\Gamma$ spans the $\mathbb{P}^3$ it is contained in, $\phi_\Gamma$ will be generally finite, and so $\dim \mathcal{F}_\Gamma = \dim \Omega_\Gamma$.

### Step 5: Proving the Theorem

Let $\Gamma \subset \mathbb{P}^n$ be a general $(n-k+2)$-plane. Note that for every line $l \in \Sigma$, $C_{\Gamma,l} = \omega_l \cap \Gamma \cap X = \Gamma \cap X_l$ is a curve in $\omega_l \cap \Gamma = \mathbb{P}^3$. Let $\mathcal{F}_\Gamma$ be the family of such curves as $l \in \Sigma$ varies. We classify $X$ based on how degenerate the curves of $\mathcal{F}_\Gamma$ are.

If every $C_{\Gamma,l}$ is a line. Then the general surface section of $X$ is ruled and so by lemma 1 from the section on Del Pezzo's theorem, $X$ is of type $R_1$, which violates our hypotheses.

Suppose now that the general curve $C_{\Gamma,l}$ spans a $\mathbb{P}^2$. If $\dim \mathcal{F}_\Gamma \geq 2$ then by proposition 6, the general surface section of $X$ is either a cone; a Veronese surface in $\mathbb{P}^4$ or $\mathbb{P}^5$; ruled; or contained in a $\mathbb{P}^3$. If the general surface section is a cone, then $X$ is a cone with vertex $\mathbb{P}^{k-2}$ and so $X$ is standard. If the general surface section is a Veronese surface then since Veronesee surfaces in $\mathbb{P}^4$ or $\mathbb{P}^5$ are not extendible, $X$ is a cone over such a surface, and so it is a scroll in $\mathbb{P}^{k-2}$. If the general surface section is ruled, then by the claim in the section on Del Pezzo's theorem, $X$ is of type $R_1$. Finally, if the general surface section is contained in $\mathbb{P}^3$, then $X \subset \mathbb{P}^{k+1}$ which violates our hypotheses.

Now suppose the general curve of $\mathcal{F}_\Gamma$ spans a 2-plane, and that $\dim \mathcal{F}_\Gamma = 1$. Let $\Lambda_{\Gamma,l} \in \mathbb{G}(2,n)$ be the 2-plane spanned by the curve $C_{\Gamma,l}$. We examine the variety

$$Z = \{(\omega, C_{\Gamma,l'}) : \Lambda_{\Gamma,l'} \subset \omega\} \subset \Omega \times \mathcal{F}_\Gamma,$$

and its projections $\pi_1$ and $\pi_2$. Since $\dim \mathcal{F}_\Gamma = 1$, $\dim \pi_2(Z) \leq 1$. Therefore, the theorem of the fibers tells us that for general $C \in \mathcal{F}_\Gamma$ and $\omega \in \Omega$, we have

$$\dim \pi_2^{-1}(C) + 1 = \dim \Omega + \dim \pi_1^{-1}(\omega) \leq 4.$$

Therefore, for the 2-plane spanned by $\Lambda$, a general curve of $\mathcal{F}_\Gamma$ we have at least a three-dimensional family of $\mathbb{P}^3$ in $\Omega_\Gamma$ containing $\Lambda$. However, we can project $X$ birationally to $\mathbb{P}^5$. When we do so, we wind up with a three-dimensional family of 3-planes in $\mathbb{P}^5$ containing a fixed 2-plane. This is a contradiction because given any 2-plane, the family of 3-planes in $\mathbb{P}^5$ containing the 2-plane is two-dimensional.

Now, suppose that the general curve in $\mathcal{F}_\Gamma$ spans a $\mathbb{P}^3$. We saw in step 4, that $\dim \mathcal{F}_\Gamma = \dim \Omega_\Gamma$, and if $X$ is nonstandard then $\dim \Omega_\Gamma \geq 4$. However, this means that the general surface section of $X$ has a 4-dimensional family of curves, each of which is contained in a $\mathbb{P}^3$. If we project the general surface section of $X$ to $\mathbb{P}^5$, we see that the the image under the projection is contained in $\mathbb{P}^4$ by proposition 5. Therefore, the general surface section is contained in a $(n-k+1)$-plane, which says that $X$ is degenerate, and so $X \subset \mathbb{P}^{n-1}$. By the same exact argument, $X \subset \mathbb{P}^{n-1}$ will be degenerate and so $X \subset \mathbb{P}^{n-2}$. Continuing in this way, we will eventually contradict the hypothesis that $X$ has codimension at least 3, thereby completing the proof of theorem 6. $\square$

## 7.13 Proof of Theorem 7

In this section we prove theorem 7, rewritten below.

**Theorem.** Let $X \subset \mathbb{P}^n$ be a $k$-dimensional, type $R_2$ variety, with $n \geq k+3$ such that for general $l \in \Sigma$, $\dim \omega_l = k+2$. Then either $X$ is standard or $X$ is a linear section of $\mathbb{G}(1,4)$.

### 7.13.1 Our Approach

As observed previously, case three is the case containing the exceptional type $R_2$ variety, $\mathbb{G}(1,4)$ (and its linear sections). Therefore, we might expect the proof of theorem 7 to be somewhat dirtier than the proofs of theorem 5 and theorem 6. We will find out soon enough that this is, indeed, the case. First, we will need the help of a slew of other classification theorems. Second, we will need to modify our techniques. Rather than dealing almost exclusively with issues such as the dimension of a family of subvarieties, and the dimension of a linear space spanned by the tangent spaces to $X$ at certain points, we will also consider issues such as the degree and the singular locus of a type $R_2$ variety satisfying case three of corollaro 1.

We prove theorem 7 first for fourfolds, and then we extend our results to varieties of any dimension. First, however, we introduce, and briefly discuss, the many classification theorems we will use.

### 7.13.2 Three Helpful Results

In this section we state the theorems which we will need to prove theorem 7. Since the proofs of most of these theorems use techniques which are vastly different than the ones we have used so far, we will not give proofs.

The first theorem we use is a classification of all varieties $X \subset \mathbb{P}^n$ such that $\deg X = n - \dim X + 1$. Such varieties are called *minimal degree varieties.* The following result is originally due to Del Pezzo, 1886 (for surfaces) and Bertini, 1907 (for higher dimensional varieties). For a proof, see [E-H].

**Proposition 7.** If $X \subset \mathbb{P}^n$ is a minimal degree variety then either

1. $X$ is a quadric hypersurface; or
2. $X$ is a cone over the Veronese surface in $\mathbb{P}^5$; or
3. $X$ is a rational normal scroll.

The next result we will use tells us the possible genus of a smooth curve of degree $d$ embedded in $\mathbb{P}^4$. In particular, the result is

**Proposition 8.** Let $C \subset \mathbb{P}^4$ be a smooth, irreducible curve of degree $d$ and genus $g$. Then if $C$ is nondegenerate

$$0 \leq g \leq \frac{1}{6}d^2 - \frac{5}{6}d + 1.$$

For a proof, see [Ra].

Our next result classifies varieties whose sectional curves rational. For a proof see [I].

**Proposition 9.** Let $X \subset \mathbb{P}^n$ be a variety of dimension $k \geq 2$ or higher such that the section of $X$ with a general $(n-k+1)$-plane is a rational curve (ie: a smooth curve of genus zero). Then

1. $X$ is a $k$-plane; or
2. $X$ is a quadric hypersurface; or
3. $X$ is a rational normal scroll; or
4. $X$ is a cone over a Veronese surface.

### 7.13.3 Proof of theorem 7 for $k = 4$

In this section, classify all nonstandard, $R_2$ fourfolds satisfying case three of corollary 1. The main result is the following.

**Theorem 8.** Let $X \subset \mathbb{P}^n$ be a fourfold of type $R_2$ with $n \geq 7$. If $X$ is such that for general $l \in \Sigma$, $\dim \omega_l = 6$, and $X$ is nonstandard then

1. $\deg X = 5$; and
2. for general $p \in X$, $X_p$ is a rational cubic cone; and
3. $X$ has elliptic sectional curves; and
4. $X$ is smooth.

*Proof.* The proof of this theorem is rather long, and so we break it up into many steps.

**Step 1: Reduction to the Case of $n = 7$**

If $X \subset \mathbb{P}^n$ is a fourfold of type $R_2$ satisfying case three of corollary 1, with $n > 7$, then we may project $X$ from $(n-7)$ general points of $\mathbb{P}^n$ to obtain a fourfold in $\mathbb{P}^7$ satisfying the hypotheses of the claim. Since degree, smoothness, and genus will all be invariant under a general projection to $\mathbb{P}^7$, the theorem for any $n \geq 7$ follows from the theorem for $n = 7$. Therefore, we proceed under the assumption that $X \subset \mathbb{P}^7$ is a type $R_2$ fourfold, satisfying case three of corollary 1.

**Step 2: Examining $X_l$**

Now, since $X$ is a type $R_2$ fourfold, for general $p \in X$ we have $\dim \Sigma_p = 1$. Therefore, for general $l \in \Sigma$ we have $2 \leq \dim X_l \leq 3$. If $\dim X_l = 2$ then a component of $X_l$ coincides with $X_p$ for general $p$, and so for general $p, q \in l$, $X_p = X_q$. This makes $X_p$ a plane, and so $X$ is standard.

So we proceed under the assumption that $\dim X_l = 3$. Notice that since $\omega_l$ is a hyperplane, $\dim(X \cap \omega_l) = 3$. Since $X_l \subset \omega_l$ we see that $X_l$ is a component of $X \cap \omega_l$. We now show that it must, in fact, be equal to $X \cap \omega_l$.

If $X$ is such that for general $l \in \Sigma$, $l \cap \omega_l$ is reducible or nonreduced, then since $\dim \Omega \geq 4$, $X$ has a four dimensional family of reducible or nonreduced hyperplane sections. By By lemma 1 from the section on Del Pezzo's theorem, this makes $X$ either type $R_1$ or a cone over a Veronese surface. Obviously, $X$ is cannot be of type $R_1$, and if $X$ is a cone over a Veronese surface, than it is $\infty^2\mathbb{P}^2$, and is standard. Therefore, we may proceed under the assumption that $X_l = X \cap \omega_l$ for general $l \in \Sigma$.

**Step 3: Examining $X \cap \omega_l \cap \omega_{l'}$**

Note first, that it cannot be that for general $p \in X$ and general $l, l' \in \Sigma_p$, we have $\omega_l = \omega_{l'}$. This is because the rational map which surjects onto a dense subset of $\Omega$,

$$\begin{aligned} \varphi: \quad & \Sigma \dashrightarrow \Omega \\ & l \mapsto \omega_l, \end{aligned}$$

must have zero-dimensional fibers since

$$4 = \dim \Sigma = \dim \Omega + \dim \varphi^{-1}(\omega) \geq 4 + \dim \varphi^{-1}(\omega).$$

Now, let $p \in X$ be a general point, and choose two general lines $l, l' \in \Sigma_p$. We examine the intersection $X \cap \omega_l \cap \omega_{l'}$. Since $\omega_l \neq \omega_{l'}$, $\dim(X \cap \omega_l \cap \omega_{l'}) = 2$. Also, note that both $\omega_l$ and $\omega_{l'}$ contain $X_p$, and so since $X_p$ is irreducible, it will be a component of the intersection.

Notice, now, that it cannot be the only component. This would mean that as $l$ and $l'$ vary in $\Sigma_p$, $X \cap \omega_l \cap \omega_{l'}$ remains constant. Since we assume $X$ is nondegenerate, this means that $\omega_l \cap \omega_{l'}$ is constant as $l$ and $l'$ vary, which we have already shown cannot be the case.

Now, we describe the other components of $X \cap \omega_l \cap \omega_{l'}$. Choose a point $x \in X \cap \omega_l \cap \omega_{l'}$. Then $x \in X_l$ and $x \in X_{l'}$. Therefore, there exists a line $s \in \Sigma_l$ containing $x$ and intersecting $l$ at the general point $q$. Similarly there exists an $s' \in \Sigma_{l'}$ containing $x$ and intersecting $l'$ at $q'$. Clearly $q, q' \in X_p$ and both $\omega_l$ and $\omega_{l'}$ contain $X_p$. Therefore, since $x, q, q' \in \omega_l \cap \omega_{l'}$ we see that $s, s' \subset \omega_l \cap \omega_{l'}$. Since there can't be infinitely many lines contained in the plane $\mathrm{Span}\{l, l'\}$ (this would mean that $X \cap \omega_l \cap \omega_{l'}$ contained the plane $\mathrm{Span}\{l, l'\}$, which would make $X$ standard), the lines $s$ and $s'$ must be elements of two distinct families of lines contained in $X \cap \omega_l \cap \omega_{l'}$. This means that every irreducible component of $X \cap \omega_l \cap \omega_{l'}$ is a quadric because it contains two families of lines. Therefore, we have

$$X \cap \omega_l \cap \omega_{l'} = X_p \cup Y_1 \cup \cdots \cup Y_N,$$

where $Y_1, \ldots, Y_N$ are quadrics containing $l$ and $l'$.

**Step 4: Proof that $N = 1$**

Consider the variety $X_l = X \cap \omega_l \subset \omega_l = \mathbb{P}^6$. For a general $p \in X_l$, there is a line $l' \in \Sigma_l$ with $p \in l'$. Then $\omega_l \cap \omega_{l'} \subset \omega_l$ is a hyperplane. If $N \geq 2$ then $p$ is a singular point of $X \cap \omega_l \cap \omega_{l'}$ with multiplicity at least 3 (since it is contained in at least three irreducible components). We take advantage of this fact by parametrizing $X_l$ locally about $p$ by

$$v(t_1, t_2, t_3) = v(T) = [v_0(T), \ldots, v_6(T)],$$

with $v(0) = p$. If $\varphi$ is the linear form cutting out the hyperplane $\omega_l \cap \omega_{l'} \subset \omega_l$, we have

$$\varphi\left(\frac{\partial^2 v}{\partial t_i \partial t_j}(0)\right) = \varphi\left(\frac{\partial v}{\partial t_i}(0)\right) = \varphi(p) = 0,$$

for every $i$ and $j$ (since $p$ has multiplicity 3). This means that

$$\mathrm{Span}\left\{\mathbb{T}_p X_l, \frac{\partial^2 v}{\partial t_i \partial t_j}\right\} \subset \omega_l \cap \omega_{l'} = \mathbb{P}^5,$$

and so $\dim V \leq 2$, where

$$V = \mathrm{Span}\left\{\frac{\partial^2 v}{\partial t_i \partial t_j}\right\} \Big/ \mathbb{T}_p X_l.$$

Since any quadric in the second fundamental form to $X_l$ at $p$, $II_p$, corresponds to an element of $V^*$, we see that $\dim II_p \leq 1$ (since $\dim II_p$ is one less than the dimension of the vector space of quadrics it spans).

Notice that if $\dim II_p = 0$ then by proposition 1, $X_l \subset Z = \infty^h\mathbb{P}^m$ where the tangent $\mathbb{P}^{h+m}$ to smooth points of the general $m$-plane are contained in a fixed $\mathbb{P}^{4-h}$. Therefore, $h+m \leq 4-h$ and so $0 \leq h \leq 2$. Clearly $h \neq 0$ since then $X_l \subset \mathbb{P}^4$ which would make it degenerate. Also, $h = 2$ forces $m = 0$ which is impossible since $\dim X_l = 3$. Finally, if $h = 1$ then $m = 2$ and so $X_l = \infty^1\mathbb{P}^2$ which would make $X$ standard.

If $\dim II_p = 1$ then also proposition 1, $X_l \subset Z = \infty^h\mathbb{P}^m$ where the tangent $\mathbb{P}^{h+m}$ to smooth points of the general $m$-plane are contained in a fixed $\mathbb{P}^{5-h}$. Again we get $0 \leq h \leq 2$. We can immediately eliminate $h = 0$ since it requires $X_l$ to be degenerate.

If $h = 1$ then $2 \leq m \leq 3$. If $m = 2$ then $X_l = \infty^1\mathbb{P}^2$ and so $X$ is standard. If $m = 3$ then we have $X_l \subset Z = \infty^1\mathbb{P}^3$. Choose a general 3-plane $\Gamma \subset Z$. Define, as before $X_\Gamma$ to be the lines contained in $X_l$ that intersect $\Gamma$. Notice that

$$X_\Gamma \subset \bigcup_{p \in X \cap \Gamma} \mathbb{T}_p X_l.$$

Clearly we have

$$X_l \cap \Gamma \subset X_\Gamma \subset X_l,$$

and so either $X_\Gamma = X_l \cap \Gamma$ or $X_\Gamma = X_l$. The latter cannot be the case since it gives

$$X_l \subset \bigcup_{p \in X \cap \Gamma} \mathbb{T}_p X_l \subset \bigcup_{p \in \Gamma} \mathbb{T}_p Z = \mathbb{P}^4.$$

Therefore, every line in $X_l$ that intersects $\Gamma$ is contained entirely within $\Gamma$. Let $s$ be a general line of $X_\Gamma$. Since $l \in \Sigma$ was general and $\Gamma \subset Z$ is general, $s$ is a general line of $\Sigma$. Therefore, since we are in case three of corollary 1, the tangent planes to $X$ along $s$ span a $\mathbb{P}^6$. Since $X_l = X \cap \omega_l$ this means that the tangent planes to $X_l$ along $s$ span a $\mathbb{P}^5$, and so they cannot be contained in the $\mathbb{P}^4$ spanned by the tangent planes to $Z$ along $\Gamma$.

Finally, if $h = 2$, $m = 1$ and so $X_l = \infty^2\mathbb{P}^1$, and the tangent planes to $X_l$ along the lines in the family span a $\mathbb{P}^3$. Since the tangent planes to $X_l$ along a general line of $\Sigma_l$ must span a $\mathbb{P}^6$ (since we are in case three of corollary 1), we see that the two dimensional family of lines covering $X_l$ is different from $\Sigma_l$. Call this family $\mathcal{F}_l$. We are interested in

$$\mathcal{F} = \bigcup_{l \in \Sigma} \mathcal{F}_l.$$

Clearly $\dim \mathcal{F} \geq 3$ since it sweeps out $X$. Suppose $\dim \mathcal{F} = 3$. By examining the variety

$$\{(l', \omega) : l' \subset \omega\} \subset \mathcal{F} \times \Omega$$

and its projections (note that $\dim \pi_2^{-1}(\omega_{l'}) = \dim \Sigma_{l'} = 2$), we see that there is a three-dimensional family of $\omega$ containing the general $l' \in \mathcal{F}$. However, by studying the variety

$$\{(p, \omega) : p \in \omega\} \subset X \times \Omega,$$

we see that there is also a three-dimensional family of $\omega$ containing the general $p \in X$. Therefore, when $p \in l'$, these families are the same and so any $\omega$ that contains $p$ will contain all of $l'$. Finally, the variety

$$\{(p, \omega) : p \in \omega\} \subset l' \times \Omega$$

tells us that every $\omega \in \Omega$ intersects $l'$. By the above this means that every $\omega \in \Omega$ contains $l'$ and so since $l' \in \mathcal{F}$ was general, we get that each $\omega \in \Omega$ contains all of $X$ which violates the nondegeneracy of $X$, since $\dim \omega = 6$. So we cannot have $\dim \mathcal{F} = 3$.

It must be, therefore, that $\dim \mathcal{F} = 4$ and $\mathcal{F} = \Sigma$. Let $\mathcal{F}_l \subset \mathcal{F}$ be the two dimensional subfamily sweeping out $X_l$. Since $\mathcal{F} = \Sigma$, for general $l \in \Sigma$, the general $l' \in \mathcal{F}_l$ will be a general line of $\Sigma$. Therefore, the fact that the family of tangent planes to $X_l$ along $l'$ spans a $\mathbb{P}^3$ violates the degeneracy of $X$. So we deduce, at last, that $N = 1$, and so

$$X \cap \omega_l \cap \omega_{l'} = X_p \cup Y_1,$$

where $Y_1$ is a quadric containing $l$ and $l'$, and $p = l \cap l'$.

**Step 5: Proving 2-4**

Notice that, not only do we have $X \cap \omega_l \cap \omega_{l'} = X_p \cup Y_1$, but we also have that $X \cap \omega_l \cap \omega_{l'}$ is reduced (ie: every component has multiplicity one), since otherwise $p$ will be a triple point and the previous argument shows that this can't be the case. Therefore, by Bézout's theorem,

$$\deg X = (\deg X)(\deg \omega_l)(\deg \omega_{l'}) = \deg X_p + \deg Y_1 = \deg X_p + 2.$$

Recall that we have already shown that if $X \subset \mathbb{P}^n$ is a $k$-dimensional, non-standard, type $R_2$ variety with $n \geq k+3$ then for general $p \in X$, $2 \leq \deg X_p \leq 3$. Therefore, we either have $\deg X = 4$ or $\deg X = 5$. Notice, finally, that if $\deg X = 4$ then $X$ is a minimal degree variety and so by proposition 7, it is either an $\infty^1\mathbb{P}^3$, a cone over a Veronese surface, or a quadric hypersurface. The first is type $R_1$, the second is standard, and the third is codimension 1. Therefore, we cannot have $\deg X = 4$ and so we must have $\deg X = 5$.

Since $\deg X_p = 3$, $X_p$ is a cone over a cubic curve contained in $\mathbb{P}^4$. Such a curve must have genus zero by proposition 8, and so we see that $X_p$ is a rational cubic cone.

Finally, since $\deg X = 5$, the section of $X$ with a general 4-plane will be a curve of degree five. By proposition 8, such a curve must have genus 0 or 1. We can exclude the genus 0 case because of the proposition 9. We conclude that the sectional curves of $X$ are elliptic curves.

**Part 6: Proving 5**

If $X$ had a singular point of multiplicity at least three, then by projecting from this point, we would obtain a fourfold of degree 2 in $\mathbb{P}^6$. This violates the lower bound for the degree of a variety $\deg X \geq 1 + \mathrm{codim} X$.

Now suppose $X$ has a double point. We can project (birationally) from the double point to obtain a degree 3 fourfold, $X' \subset \mathbb{P}^6$. Then $X'$ is a minimal degree variety and so by proposition 7, either it is a quadric hypersurface, a cone over a Veronese surface, or a rational normal scroll. The quadric hypsurface and the cone over the Veronese surface are not of degree 3, and so we must have $X' = \infty^1\mathbb{P}^3$. Let $\Gamma \subset X'$ be a general three plane, and let $X_\Gamma$ be the preimage of $\Gamma$ under the projection. We calculate $\deg X_\Gamma$.

Clearly $\deg X_\Gamma \neq 1$ since then $X$ would be $\infty^1\mathbb{P}^3$ and so it wouldn't be of type $R_2$. Let $Y \subset \mathbb{P}^7$ be a quadric hypersurface vanishing on $X$. Let $\Lambda$ be the 4-plane spanned by $X_\Gamma$ (ie: the 4-plane spanned by $\Gamma$ and the point of projection). Clearly $X_\Gamma \subset Y$, and if $\deg X_\Gamma \geq 3$ then $\Lambda \subset Y$, since otherwise we would have

$$2 = (\deg Y)(\deg H) \geq \deg X_\Gamma \geq 3.$$

But this means that $X$ is not a component of the quadrics that contain it, which is a contradiction. Therefore, we must have $\deg X_\Gamma = 2$, and so $X$ is a one-parameter family of quadrics. However, we assumed $X$ was nonstandard and so it must be that $X$ is smooth, as desired. □

**Note.** Due to the classification of polarized, smooth varieties given in [I], we see that if $X \subset \mathbb{P}^n$ is a four-dimensional, type $R_2$ variety, with $n \geq 7$, and satisfying case three of corollary 1, then $X$ is standard or it is the intersection of $\mathbb{G}(1,4)$ with two nontangent hyperplanes.

**Note.**

### 7.13.4 Proof of Theorem 7

Finally, we conclude the proof to theorem 7, which completes the proof of our classification theorem for type $R_2$ varieties with codimension greater than 2.

**Claim.** Let $X \subset \mathbb{P}^n$ be a $k$-dimensional variety of type $R_2$ with $n \geq k+3$ satisfying case three of corollary 1. Then either $X$ standard, or $X$ is $\mathbb{G}(1,4)$ of a smooth linear section of $\mathbb{G}(1,4)$.

*Proof.* Such an $X$ will be such that its general 4-dimensional section will satisfy the hypotheses of theorem 8. Therefore, either its 4-dimensional section is a smooth extension of the Scorza variety, or it is standard, which says exactly that either $X$ is standard or else it is $\mathbb{G}(1,4)$ or a smooth linear section of $\mathbb{G}(1,4)$. □

**Note.** This completes the proof of theorem 7, which in turn, completes the proof of theorem 3.

# 8 Reflection

This theorem raises many questions, and in this section we discuss some of them.

## 8.1 The Difficulty of Codimension 2

Maybe the first batch of questions that comes to mind are ones similar to 'Why is this approach not able to say anything about codimension 1 or 2?' or 'Where exactly does the argument rely on the codimension being larger than 2?' The answer to this is in the way we used the second fundamental form. In some sense we began with the observation that if a variety contains many lines, then the base locus of the second fundamental form at a point will have large dimension (since the variety swept out by the lines through a point are contained inside the base locus). A large dimensional base locus gives rise to a low dimensional second fundamental form, and we were able to exploit this feature of type $R_2$ varieties. However, low dimensional second fundamental forms also come about via low codimension. Indeed, the second fundamental form is the image of the dual of the normal space, and so if codimension is low, the normal space will be small which will yield a low dimensional second fundamental form. Therefore, the smaller the codimension, the less we were able to distinguish between a variety with many lines and an arbitrary variety, and so it is no surprise that our methods did not permit us to sufficiently handle this case.

## 8.2 What Can be said for Codimension $\leq 2$?

First, notice that we do have at least two more examples of type $R_2$ varieties if we make no restriction on the codimension. The first is the intersection of two quadratic hypersurfaces. To see that such a variety is type $R_2$ we examine the variety

$$\Phi = \{(Y, Y', l) : l \in Y \cap Y'\} \subset \mathbb{P}^N \times \mathbb{P}^N \times \mathbb{G}(1, n),$$

where $\mathbb{P}^N$ is the projective space parametrizing all quadric hypersurfaces in $\mathbb{P}^n$. Let $\pi_1 : Z \to \mathbb{P}^N \times \mathbb{P}^N$ be the projection onto the first two components and let $\pi_2 : Z \to \mathbb{G}(1, n)$ be projection onto the third. Without too much difficulty we could show that $\pi_1$ is surjective, and so we have

$$\dim F_1(Y \cap Y') = 2(n-1) + 2(N-3) - 2N = 2n - 8 = 2(n-2) - 4,$$

and so $Y \cap Y'$ is type $R_2$.

Another example of a type $R_2$ variety is a cubic hypersurface. To see that this is of type $R_2$ we define the variety

$$\Phi = \{(Z, l) : l \in Z\} \subset \mathbb{P}^M \times \mathbb{G}(1, n),$$

where $\mathbb{P}^M$ is the projective space parametrizing all cubic hypersurfaces. Let $\pi_1$ and $\pi_2$ be the projections. This time it is a bit harder to show that $\pi_1$ is surjective, however it is the case (see, for example, Alex Waldron's thesis), and so we have

$$\dim F_1(Z) = 2(n-1) + (N-4) - N = 2(n-1) - 4,$$

and so the general cubic hypersurface is type $R_2$ also.

In 2005, Landsberg and Robles showed that if a type $R_2$ variety has the property that for a general point $p$, every tangent line intersecting the variety at $p$ does so with multiplicity at least 3 (this is called the Fubini hypothesis), then the variety is one of the five which we have already seen.

## 8.3 What About Varieties of Type $R_3$?

I do believe it would be possible to apply similar results as we did to be able to say something about varieties of type $R_3$. However, at least two problems would arise. First, as varieties of type $R_3$ contain slightly less lines than those of type $R_2$, we would expect the dimension of the second fundamental form to be slightly higher, which means that in order to realize a distinction between type $R_3$ varieties and arbitrary varieties, we would be forced to sacrifice a little bit more codimension. While codimension at least 3 or 4 (which is what we would likely have to settle for) is still a positive result, at some point, this technique will stop being useful.

Another problem which we would likely encounter is an increased number of counterexamples. For our $R_2$ result, we only had the one exceptional type $R_2$, $\mathbb{G}(1,4)$, to worry about. We split up the theorem into three cases and we examined each one individually. In both cases that were exception-free, our second fundamental form related analysis worked perfectly. However, to deal with the exceptional case, we basically dropped the second fundamental form approach and we were forced to hack around dealing with issues such as degree and smoothness and genus, and referring to all sorts of preexisting classification theorems. If we were to try to apply these techniques to deal with the type $R_3$ problem, it is likely that still more exceptions would arise and we would be forced to appeal to this ad hoc form of argument much more consistently than in the $R_2$ problem.

## General References

[H] J. Harris, *Algebraic Geometry: a First Course*, Springer-Verlag, 1992.

[Hart] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, 1977.

[Shaf] I. Shafarevich, *Basic Algebraic Geometry: Varieties in Projective Space* Pt 1, Springer Verlag.

## Specific References

[E-H] D. Eisenbud, J. Harris, *On Varieties of Minimal Degree*, Algebraic Geometry Bowdoin 1985, Vol. 46 pt 1.

[G-H] P. Griffiths, J. Harris, *Algebraic geometry and local differential geometry*, Ann. Sci. Ecole Norm. Sup. 12 (1979), 99-106.

[I] P. Ionescu, *Projective varieties of small invariants*, Proceedings of the week of algebraic geometry, Bucharest 1982 (L.N.M. vol 1056), Berlin, Heeidelberg, New York Springer 1984.

[L-R] J.M. Landsberg, C. Robles, *Fubini's Theorem in codimension two* (2005) www.math.tamu.edu/jml/

[Me] E. Mezzetti, *Projectivee varieties with many degenerate subvarieties*, preprint.

[Ra] J. Rathmann, *The Genus of Curves in* $\mathbb{P}^4$ *and* $\mathbb{P}^5$, Mathematische Zeitschrift, Springer-Verlag 1989.

[R1] E. Rogora, *Varieties with many Lines*, Manuscripta Math. 82 (1994), no. 2, 207-226.

[R2] E. Rogora, *Metodi proiettivi per lo studio de alcune questioni relative alla varieta immerse*, tesi per il conseguimento del titolo de Dottore de Ricera, IV ciclo.

[S] B. Segre, *Sulle* $V_n$ *aventi piu de* $\infty^{n-k} S_k$, note I e II, Rendiconti dell'accademia nazionale dei Lincei, 1948, vol. V.

[T] A. Terracini, *Intorno ai punti doppi impropri etc.* Rend. circ. mat. Palermo 15, 1901, pp. 33-51, n.10.

[Z] F. Zak, *Tangents and secants of algebraic varieties*, Translations of Mathematical Monographs 127, AMS 1993.