THE MODULI STACK OF G-BUNDLES

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Contents

1. INTRODUCTION

Let X be a smooth projective curve over an algebraically closed field k of characteristic 0. Then the moduli stack Bun_r is defined to be the k-stack whose fiber over a k-scheme T consists of the groupoid of vector bundles, or equivalently locally free sheaves, of rank r on $X \times T$. The geometric Langlands program studies the correspondence between $\mathcal{D}$-modules on Bun_r and local systems on X, and this correspondence has deep connections to both number theory and quantum physics (see [Fre10]). It is therefore important to understand the moduli problem of vector bundles on families of curves. As vector bundles of rank r correspond to GL_r-bundles, we naturally extend our study to G-bundles for an algebraic group G. Since G-bundles may have very large automorphism groups, the natural setting for understanding their moduli problem is the moduli stack (as opposed to the coarse moduli space).

Let $X \to S$ be a morphism of schemes over an arbitrary base field k. We focus our attention on the S-stack Bun_G whose fiber over an S-scheme T consists of the groupoid of principal

G-bundles over $X \times_S T$. Artin's notion of an algebraic stack provides a nice framework for our study of Bun_G. The goal of this paper is to give an expository account of the geometric properties of the moduli stack Bun_G. The main theorem concerning Bun_G is the following:

Theorem 1.0.1. *For a flat, finitely presented, projective morphism $X \to S$, the S-stack Bun_G is an algebraic stack locally of finite presentation over S, with a schematic, affine diagonal of finite presentation. Additionally, Bun_G admits an open covering by algebraic substacks of finite presentation over S.*

The proof that Bun_G is algebraic has been known to experts for some time, and the result follows from [Ols06] (see also [Beh91], [Bro10], and [Sor00]). To our knowledge, however, there is no complete account in the literature of the proof we present, which gives more specific information for the case of an algebraic group G.

We now summarize the contents of this paper. In §2, we introduce the stack quotient $[Z/G]$ of a k-scheme Z by a G-action, which is a fundamental object for what follows. We show that these quotients are algebraic stacks, and we discuss the relations between two different quotients. In order to study Bun_G, we prove several properties concerning Hom stacks in §3. The key result in this section is that the sheaf of sections of a quasi-projective morphism $Y \to X$ is representable over S. Theorem 1.0.1 is proved in §4. In §5, we define a level structure on Bun_G using nilpotent thickenings of an S-point of X. This provides an alternative presentation of a quasi-compact open substack of Bun_G. Lastly in §6, we prove that Bun_G is smooth over S if G is smooth over k and $X \to S$ has fibers of dimension 1.

1.1. **Acknowledgments.** I would like to thank my advisor, Dennis Gaitsgory, for introducing and teaching this beautiful subject to me. My understanding of what is written here results from his invaluable support and guidance. I also wish to thank Thanos D. Papaïoannou and Simon Schieder for many fruitful discussions.

1.2. **Notation and terminology.** We fix a field k, and we will mainly work in the category $\mathbf{Sch}_{/k}$ of schemes over k. For two k-schemes X and Y, we will use $X \times Y$ to denote $X \times_{\mathrm{Spec}\, k} Y$. We will consider a scheme as both a geometric object and a sheaf on the site $(\mathbf{Sch})_{\mathrm{fpqc}}$ without distinguishing the two. For k-schemes X and S, we denote $X(S) = \mathrm{Hom}_k(S, X)$. For two morphisms of schemes $X \to S$ and $Y \to S$, we use $\mathrm{pr}_1 : X \times_S Y \to X$ and $\mathrm{pr}_2 : X \times_S Y \to Y$ to denote the projection morphisms when there is no ambiguity. Suppose we have a morphism of schemes $p : X \to S$ and an $\mathcal{O}_X$-module $\mathcal{F}$ on X. For a morphism $T \to S$, we will use the notation $X_T = X \times_S T$ and $p_T = \mathrm{pr}_2 : X_T \to T$, which will be clear from the context. In the same manner, we denote $\mathcal{F}_T = \mathrm{pr}_1^* \mathcal{F}$ (this notation also applies when $X = S$). For a point $s \in S$, we let $\kappa(s)$ equal the residue field. Then using the previous notation, we call the fiber $X_s = X_{\mathrm{Spec}\, \kappa(s)}$ and $\mathcal{F}_s = \mathcal{F}_{\mathrm{Spec}\, \kappa(s)}$. For a locally free $\mathcal{O}_X$-module $\mathcal{E}$ of finite rank, we denote the dual module $\mathcal{E}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$, which is also locally free of finite rank.

We will use Grothendieck's definitions of quasi-projective [EGAII, 5.3] and projective [EGAII, 5.5] morphisms. That is, a morphism $X \to S$ of schemes is quasi-projective if it is of finite type and there exists a relatively ample invertible sheaf on X. A morphism $X \to S$ is projective if there exists a quasi-coherent $\mathcal{O}_S$-module $\mathcal{E}$ of finite type such that X is S-isomorphic to a closed subscheme of $\mathbf{P}(\mathcal{E})$. We say that $X \to S$ is locally quasi-projective (resp. locally projective) if there exists an open covering $(U_i \subset S)$ such that the morphisms $X_{U_i} \to U_i$ are quasi-projective (resp. projective). Following [AK80], a morphism $X \to S$ is strongly quasi-projective (resp. projective) if it is finitely presented and there exists a locally free $\mathcal{O}_S$-module $\mathcal{E}$ of constant finite rank such that X is S-isomorphic to a retrocompact (resp. closed) subscheme of $\mathbf{P}(\mathcal{E})$.

We will use G to denote an algebraic group over k, by which we mean an affine group scheme of finite type over k (note that we do not require G to be reduced). For a k-scheme S, we say that a sheaf $\mathcal{P}$ on $(\mathbf{Sch}_{/S})_{\mathrm{fpqc}}$ is a right G-bundle over S if it is a right $G|_S$-torsor (see [DG70], [Gir71] for basic facts on torsors). Since affine morphism of schemes are effective under fpqc descent [FGI$^+$05, Theorem 4.33], the G-bundle $\mathcal{P}$ is representable by a scheme affine over S. As $G \to \operatorname{Spec} k$ is fppf, $\mathcal{P}$ is in fact locally trivial in the fppf topology. If G is smooth over k, then $\mathcal{P}$ is locally trivial in the étale topology.

For a sheaf of groups $\mathcal{G}$ on a site $\mathcal{C}$ and $S \in \mathcal{C}$, we will use $g \in \mathcal{G}(S)$ to also denote the automorphism of $\mathcal{G}|_S$ defined by left multiplication by g. This correspondence gives an isomorphism of sheaves $\mathcal{G} \simeq \operatorname{Isom}(\mathcal{G}, \mathcal{G})$, where the latter consists of right $\mathcal{G}$-equivariant morphisms.

We will use pseudo-functors, instead of fibered categories, in our formal setup of descent theory. The reasoning is that this makes our definitions more intuitive, and pseudo-functors are the 2-category theoretic analogue of presheaves of sets. See [FGI$^+$05, 3.1.2] for the correspondence between pseudo-functors and fibered categories. By a stack we mean a pseudo-functor satisfying the usual descent conditions. We will implicitly use the 2-Yoneda lemma, i.e., for a pseudo-functor $\mathcal{Y}$ and a scheme S, an element of $\mathcal{Y}(S)$ corresponds to a morphism $S \to \mathcal{Y}$. We say that a morphism $\mathcal{X} \to \mathcal{Y}$ of pseudo-functors is representable (resp. schematic) if for any scheme S mapping to $\mathcal{Y}$, the 2-fibered product $\mathcal{X} \times_{\mathcal{Y}} S$ is isomorphic to an algebraic space (resp. a scheme).

We use the definition of an algebraic stack given in [Sta]: an algebraic stack $\mathcal{X}$ over a scheme S is a stack in groupoids on $(\mathbf{Sch}_{/S})_{\mathrm{fppf}}$ such that the diagonal $\mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable and there exists a scheme U and a smooth surjective morphism $U \to \mathcal{X}$. We call such a morphism $U \to \mathcal{X}$ a presentation of $\mathcal{X}$. Note that this definition is weaker than the one in [LMB00] as there are less conditions on the diagonal. We will use the definitions of properties of algebraic stacks and properties of morphisms between algebraic stacks from [LMB00] and [Sta].

2. QUOTIENT STACKS

In this section, fix a k-scheme Z with a right G-action $\alpha : Z \times G \to Z$. All schemes mentioned will be k-schemes.

Definition 2.0.1. The stack quotient $[Z/G]$ is the pseudo-functor $(\mathbf{Sch}_{/k})^{\mathrm{op}} \to \mathbf{Gpd}$ with

$$[Z/G](S) = \{\text{Right } G\text{-bundle } \mathcal{P} \to S \text{ and } G\text{-equivariant morphism } \mathcal{P} \to Z\}$$

where a morphism from $\mathcal{P} \to Z$ to $\mathcal{P}' \to Z$ is a G-equivariant morphism $\mathcal{P} \to \mathcal{P}'$ over $S \times Z$.
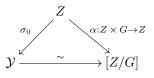For $\cdot = \operatorname{Spec} k$ with the trivial G-action, we call $BG = [\cdot/G]$.

By considering all of our objects as sheaves on $(\mathbf{Sch}_{/k})_{\mathrm{fpqc}}$, we observe that $[Z/G]$ is an fpqc stack. The main result of this section is that $[Z/G]$ is an algebraic stack:

Theorem 2.0.2. *The k-stack $[Z/G]$ is an algebraic stack with a schematic, separated diagonal. If Z is quasi-separated, then the diagonal $\Delta_{[Z/G]}$ is quasi-compact. If Z is separated, then $\Delta_{[Z/G]}$ is affine.*

Remark 2.0.3. In fact, the proof will show that Theorem 2.0.2 holds if G is a group scheme affine, flat, and of finite presentation over a base scheme S and Z is an S-scheme with a right G-action, in which case $[Z/G]$ is an algebraic S-stack. We choose to work over the base $\operatorname{Spec} k$ in this section because some results in §2.4 do not hold in greater generality.

2.1. **Characterizing** $[Z/G]$**.** It will be useful in the future to know when a stack satisfying some properties is isomorphic to $[Z/G]$. In particular, we show the following (the conditions for a morphism from a scheme to a stack to be a G-bundle will be made precise later).

Lemma 2.1.1. *Suppose $\mathcal{Y}$ is a k-stack and $\sigma_0 : Z \to \mathcal{Y}$ is a G-bundle. Then there is an isomorphism $\mathcal{Y} \to [Z/G]$ of stacks such that the triangle*
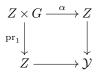
$$
\begin{array}{ccc}
 & Z & \\
\sigma_0 \swarrow & & \searrow \alpha : Z \times G \to Z \\
\mathcal{Y} & \xrightarrow{\ \sim\ } & [Z/G]
\end{array}
$$

is 2-commutative.

Remark 2.1.2. Lemma 2.1.1 in particular shows that if X is a scheme and $Z \to X$ is a G-bundle over X, then $X \simeq [Z/G]$. Therefore these two notions of a quotient, as a scheme and as a stack, coincide.

The rest of this subsection will be slightly technical. We first define the notion of a G-invariant morphism to a stack, which then naturally leads to the definition of a morphism being a G-bundle. We then introduce two lemmas, which we use to prove Lemma 2.1.1.

2.1.3. *G-invariant morphisms to a stack.* Let $\mathcal{Y}$ be a k-stack and $\sigma_0 : Z \to \mathcal{Y}$ a morphism of stacks. We say σ_0 is *G-invariant* if the following conditions are satisfied:

(1) The diagram

$$
\begin{array}{ccc}
Z \times G & \xrightarrow{\ \alpha\ } & Z \\
\mathrm{pr}_1 \downarrow & & \downarrow \\
Z & \longrightarrow & \mathcal{Y}
\end{array}
$$

is 2-commutative. This is equivalent to having a 2-morphism $\rho : \mathrm{pr}_1^* \sigma_0 \to \alpha^* \sigma_0$.
(2) For a scheme S, if we have $z \in Z(S)$ and $g \in G(S)$, then we let $\rho_{z,g}$ denote the corresponding 2-morphism

$$
z^* \sigma_0 \simeq (z,g)^* \mathrm{pr}_1^* \sigma_0 \xrightarrow{(z,g)^* \rho} (z,g)^* \alpha^* \sigma_0 \simeq (z.g)^* \sigma_0.
$$

The $\rho_{z,g}$ must satisfy an associativity condition. More precisely, for $z \in Z(S)$ and $g_1, g_2 \in G(S)$, we require

$$
\begin{array}{ccc}
z^* \sigma_0 & \xrightarrow{\ \rho_{z,g_1}\ } & (z.g_1)^* \sigma_0 \\
\rho_{z,g_1 g_2} \downarrow & & \downarrow \rho_{z.g_1,g_2} \\
(z.(g_1 g_2))^* \sigma_0 & =\!=\!= & ((z.g_1).g_2)^* \sigma_0
\end{array}
$$

to commute.

For a general treatment of group actions on stacks, we refer the reader to [Rom05].

Suppose that $\sigma_0 : Z \to \mathcal{Y}$ is a G-invariant morphism of stacks. For any scheme S with a morphism $\sigma : S \to \mathcal{Y}$, the 2-fibered product $Z \times_{\mathcal{Y}} S$ is a sheaf of sets. We show that $Z \times_{\mathcal{Y}} S$ has an induced right G-action such that $Z \times_{\mathcal{Y}} S \to S$ is G-invariant and $Z \times_{\mathcal{Y}} S \to Z$ is G-equivariant. By definition of the 2-fibered product, for a scheme T we have

$$(2.1.3.1) \qquad \left(Z \times_{\mathcal{Y}} S \right)(T) = \left\{ (a,b,\phi) \mid a \in Z(T), b \in S(T), \phi : a^* \sigma_0 \to b^* \sigma \right\}.$$

We define a right G-action on $Z \times_{\mathcal{Y}} S$ by letting $g \in G(T)$ act via
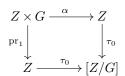
$$(a, b, \phi).g = (a.g, b, \phi \circ \rho_{a,g}^{-1}).$$

Since $\rho_{a.g_1, g_2} \circ \rho_{a, g_1} = \rho_{a, g_1 g_2}$, this defines a natural G-action. From this construction, it is evident that $Z \times_{\mathcal{Y}} S \to S$ is G-invariant and $Z \times_{\mathcal{Y}} S \to Z$ is G-equivariant.

2.1.4. *G-bundles over a stack.* Let $\mathcal{Y}$ be a stack and $\sigma_0 : Z \to \mathcal{Y}$ a G-invariant morphism of stacks. We say that $Z \to \mathcal{Y}$ is a *G-bundle* if for any scheme S mapping to $\mathcal{Y}$, the induced G-action on $Z \times_{\mathcal{Y}} S \to S$ gives a G-bundle. In particular, σ_0 is schematic.
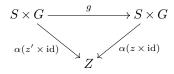
2.1.5. Let $\tau_0 \in [Z/G](Z)$ correspond to the trivial G-bundle $\mathrm{pr}_1 : Z \times G \to Z$ with the G-equivariant morphism $\alpha : Z \times G \to Z$.

Lemma 2.1.6. *The diagram*

$$
\begin{array}{ccc}
Z \times G & \xrightarrow{\ \alpha\ } & Z \\
{\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \tau_0} \\
Z & \xrightarrow{\ \tau_0\ } & [Z/G]
\end{array}
$$

is a 2-Cartesian square, and τ_0 is a G-invariant morphism.

Proof. Suppose for a scheme S we have $z, z' \in Z(S)$ and $\phi : z'^* \tau_0 \simeq z^* \tau_0$. Then ϕ corresponds to $g \in G(S)$ such that

$$
\begin{array}{ccc}
S \times G & \xrightarrow{\ \ \ g\ \ \ } & S \times G \\
 & {\scriptstyle \alpha(z' \times \mathrm{id})}\searrow \quad \swarrow {\scriptstyle \alpha(z \times \mathrm{id})} & \\
 & Z &
\end{array}
$$

commutes. This implies that $z.g = z'$. Conversely, $z \in Z(S)$ and $g \in G(S)$ uniquely determine $z' = z.g$ and a G-equivariant morphism $z'^* \tau_0 \to z^* \tau_0$ corresponding to g. Therefore the morphism

$$(\alpha, \mathrm{pr}_1) : Z \times G \to Z \underset{[Z/G]}{\times} Z$$

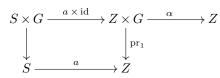is an isomorphism. Taking $S = Z \times G$ and $\mathrm{id}_{Z \times G}$ gives an isomorphism

$$\rho^{-1} : \alpha^*(\tau_0) = (\mathrm{pr}_1 . \mathrm{pr}_2)^*(\tau_0) \to \mathrm{pr}_1^*(\tau_0),$$

which is defined explicitly by $Z \times G \times G \to Z \times G \times G : (z, g_1, g_2) \mapsto (z, g_1, g_1 g_2)$. From this we see that for a scheme S and $z \in Z(S), g \in G(S)$, the morphism $\rho_{z,g}$ corresponds to the morphism of schemes $S \times G \to S \times G : (s, g_0) \mapsto (s, g(s)^{-1} g_0)$. Therefore $g_2^{-1} g_1^{-1} = (g_1 g_2)^{-1}$ implies τ_0 is a G-invariant morphism. $\square$
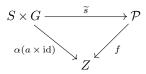
Remark 2.1.7. Now that we know τ_0 is G-invariant, $Z \times_{[Z/G]} Z$ has an induced G-action as a sheaf of sets. The morphism $(\alpha, \mathrm{pr}_1) : Z \times G \to Z \times_{[Z/G]} Z$ sends $(z, g) \mapsto (z.g, z, \rho_{z,g}^{-1})$ on S-points, so the associativity condition on ρ implies that (α, pr_1) is a G-equivariant morphism of sheaves of sets.

Lemma 2.1.8. *Let $\tau = (f : \mathcal{P} \to Z) \in [Z/G](S)$ for a scheme S, and suppose $\mathcal{P}$ admits a section $s \in \mathcal{P}(S)$. Then the G-equivariant morphism $\widetilde{s} : S \times G \to \mathcal{P}$ induced by s gives an isomorphism $(f \circ s)^* \tau_0 \to \tau \in [Z/G](S)$.*

Proof. Let $a = f \circ s : S \to \mathcal{P} \to Z$. We have a Cartesian square

$$
\begin{array}{ccc}
S \times G & \xrightarrow{\ a \times \mathrm{id}\ } Z \times G \xrightarrow{\ \alpha\ } Z \\
\downarrow & \quad\ \ \downarrow{\scriptstyle \mathrm{pr}_1} \\
S & \xrightarrow{\ \ \ a\ \ \ } Z
\end{array}
$$

so $a^* \tau_0 = (\alpha(a \times \mathrm{id}) : S \times G \to Z)$. The diagram

$$
\begin{array}{ccc}
S \times G & \xrightarrow{\ \ \ \widetilde{s}\ \ \ } & \mathcal{P} \\
 & {\scriptstyle \alpha(a \times \mathrm{id})} \searrow \quad \swarrow {\scriptstyle f} & \\
 & Z &
\end{array}
$$

is commutative by G-equivariance, since the morphisms agree after composing by the section $S \to S \times G$ corresponding to $1 \in G(S)$. Therefore $\widetilde{s}$ is a morphism in $[Z/G](S)$. $\qquad\square$

We are now ready to prove the claimed characterization of $[Z/G]$.

Proof of Lemma 2.1.1. We define the morphism $F : \mathcal{Y} \to [Z/G]$ by sending $\sigma \in \mathcal{Y}(S)$ to

$$
F(\sigma) = \begin{array}{ccc}
Z \underset{\mathcal{Y}}{\times} S & \xrightarrow{\hspace{2cm}} & Z \\
\downarrow & & \vdots {\scriptstyle \sigma_0} \\
S & \dashrightarrow[\sigma] & \mathcal{Y}
\end{array}
$$

for any scheme S. Since $Z \to \mathcal{Y}$ is a G-bundle, $Z \times_{\mathcal{Y}} S \to Z$ is indeed an object of $[Z/G](S)$ (see §2.1.4). For $\sigma, \sigma' \in \mathcal{Y}(S)$, let $\mathcal{P} = Z \times_{\mathcal{Y},\sigma} S$ and $\mathcal{P}' = Z \times_{\mathcal{Y},\sigma'} S$ be the fibered products. We have $\mathcal{P}(T) = \{(a, b, \phi)\}$ following the notation of (2.1.3.1). A morphism $\psi : \sigma \to \sigma'$ induces a G-equivariant morphism $\mathcal{P} \to \mathcal{P}'$ over $S \times Z$ by sending

$$(2.1.8.1) \qquad\qquad\qquad (a, b, \phi) \mapsto (a, b, b^* \psi \circ \phi).$$

This endows F with the structure of a morphism of stacks $\mathcal{Y} \to [Z/G]$.

We will first show 2-commutativity of the triangle. Then we prove that F is fully faithful and essentially surjective to deduce that it is an isomorphism.

Step 1. Showing the triangle is 2-commutative is equivalent to giving a morphism $\tau_0 \to F(\sigma_0)$ in $[Z/G](Z)$. The discussion of Remark 2.1.7 shows that

$$(\alpha, \mathrm{pr}_1) : Z \times G \to Z \underset{\mathcal{Y}}{\times} Z$$

gives a G-equivariant morphism of sheaves over $Z \times Z$, and this gives the desired morphism $\tau_0 \to F(\sigma_0)$ in $[Z/G](Z)$.

Step 2. We show that F is fully faithful. Since $\mathcal{Y}, [Z/G]$ are stacks, F is fully faithful if and only if for any fixed scheme S and $\sigma, \sigma' \in \mathcal{Y}(S)$, the induced morphism of sheaves of sets

$$F : \mathrm{Isom}_{\mathcal{Y}(S)}(\sigma, \sigma') \to \mathrm{Isom}_{[Z/G](S)}(F(\sigma), F(\sigma'))$$

on $\mathbf{Sch}_{/S}$ is an isomorphism. Let $\mathcal{P} = Z \times_{\mathcal{Y},\sigma} S$ and $\mathcal{P}' = Z \times_{\mathcal{Y},\sigma'} S$. By restricting to an fppf covering $(S_i \to S)$, we reduce to the case where $\mathcal{P}, \mathcal{P}'$ are both trivial bundles. Let $\Psi : \mathcal{P} \to \mathcal{P}'$ be a G-equivariant morphism over $S \times Z$. Since $\mathcal{P}$ is trivial, there is a section $s \in \mathcal{P}(S)$ corresponding to $(a, \mathrm{id}_S, \phi : a^* \sigma_0 \to \sigma)$ with $a \in Z(S)$. Then Ψ sends s to some element $(a, \mathrm{id}_S, \phi' : a^* \sigma_0 \to \sigma') \in \mathcal{P}'(S)$. We define a morphism of sets

$$L : \mathrm{Hom}_{[Z/G](S)}(F(\sigma), F(\sigma')) \to \mathrm{Hom}_{\mathcal{Y}(S)}(\sigma, \sigma').$$

by $L(\Psi) = \phi' \circ \phi^{-1} : \sigma \to \sigma'$.

It remains to check that F, L are mutually inverse (here by F we really mean the morphism F_S of Hom sets). Starting with $\psi : \sigma \simeq \sigma'$, we have from (2.1.8.1) that $F(\psi)$ sends s to $(a, \mathrm{id}_S, \psi \circ \phi) \in \mathcal{P}'(S)$. Thus $LF(\psi) = (\psi \circ \phi) \circ \phi^{-1} = \psi$. If we instead start with $\Psi : \mathcal{P} \to \mathcal{P}'$, then $L(\Psi) = \phi' \circ \phi^{-1}$ induces

$$FL(\Psi) : s \mapsto (a, \mathrm{id}_S, (\phi' \circ \phi^{-1}) \circ \phi) \in \mathcal{P}'(S)$$

again by (2.1.8.1). A G-equivariant morphism of trivial bundles $\mathcal{P} \to \mathcal{P}'$ over S is determined by the image of the section $s \in \mathcal{P}(S)$; hence $\Psi = FL(\Psi)$. We conclude that F is fully faithful.

Step 3. We now prove that F is essentially surjective and hence an isomorphism of stacks. Let $\tau = (f : \mathcal{P} \to Z) \in [Z/G](S)$. Choose an fppf covering $(j_i : S_i \to S)$ trivializing $\mathcal{P}$. Then we have sections $s_i \in \mathcal{P}(S_i)$. Let f_i denote the restriction of f to $\mathcal{P}_{S_i}$. Lemma 2.1.8 gives an isomorphism $(f_i \circ s_i)^* \tau_0 \simeq j_i^* \tau$. We already know that $\tau_0 \simeq F(\sigma_0)$ from Step 1. Therefore

$$F((f_i \circ s_i)^* \sigma_0) \simeq (f_i \circ s_i)^* F(\sigma_0) \simeq (f_i \circ s_i)^* \tau_0 \simeq j_i^* \tau.$$

By [Sta, Lemma 046N], we conclude that F is an isomorphism. $\qquad\square$

2.2. **Twisting by torsors.** In this subsection, we review a construction that will come up in many places. Let $\mathcal{C}$ be a subcanonical site with a terminal object and a sheaf of groups $\mathcal{G}$. For $S \in \mathcal{C}$, let $\mathcal{P}$ be a right $\mathcal{G}|_S$-torsor over S. Suppose we have a sheaf of sets $\mathcal{F}$ on $\mathcal{C}$ with a left $\mathcal{G}$-action. Then $\mathcal{G}|_S$ acts on $\mathcal{P} \times \mathcal{F}$ from the right by $(p, z).g = (p.g, g^{-1}.z)$. We have a presheaf $\mathcal{Q}$ on $\mathcal{C}/S$ defined by taking $\mathcal{G}$-orbits $\mathcal{Q}(U) = (\mathcal{P}(U) \times \mathcal{F}(U))/\mathcal{G}(U)$. We define the sheaf

$$_{\mathcal{P}}\mathcal{F} = (\mathcal{P} \times \mathcal{F})/\mathcal{G} = \mathcal{P} \overset{\mathcal{G}}{\times} \mathcal{F}$$

obtained by twisting $\mathcal{F}$ by $\mathcal{P}$ to be the sheafification of $\mathcal{Q}$.

Since sheaves on $\mathcal{C}$ form a stack, we give an alternative description of $_{\mathcal{P}}\mathcal{F}$ by providing a descent datum. Let $(S_i \to S)$ be a covering such that $\mathcal{P}|_{S_i} \simeq \mathcal{G}|_{S_i}$. Then $(\mathcal{G}|_{S_i}, g_{ij})$ give a descent datum of $\mathcal{P}$, for some $g_{ij} \in \mathcal{G}(S_i \times_S S_j)$. Note that

$$(\mathcal{P}|_{S_i} \times \mathcal{F})/\mathcal{G} \simeq (\mathcal{G}|_{S_i} \times \mathcal{F})/\mathcal{G} \simeq \mathcal{F}|_{S_i},$$

and $\mathcal{Q}|_{S_i} \simeq \mathcal{F}|_{S_i}$ is already a sheaf. By the definition of the group action on $\mathcal{P} \times \mathcal{F}$, we see that the transition morphism $\varphi_{ij} : (\mathcal{F}|_{S_j})|_{S_i \times_S S_j} \to (\mathcal{F}|_{S_i})|_{S_i \times_S S_j}$ is given by the left action of g_{ij}. Since sheafification commutes with the restriction $\mathcal{C}/S_i \to \mathcal{C}/S$, we conclude that

$$(\mathcal{F}|_{S_i}, \varphi_{ij})$$

gives a descent datum for $_{\mathcal{P}}\mathcal{F}$.

If $\mathcal{F}$ is instead a sheaf with a right $\mathcal{G}$-action, we will use $_{\mathcal{P}}\mathcal{F}$ to denote the twist of $\mathcal{F}$ considered with the inverse left $\mathcal{G}$-action.

We will use the above twisting construction for the big site $\mathcal{C} = (\mathbf{Sch}_{/k})_{\mathrm{fppf}}$. Let V be a G-representation. Then V can be considered as an abelian fppf sheaf via pullback (see [Sta, Lemma 03DT]) with a left G-action. Then for a right G-bundle $\mathcal{P}$ over S, we have a quasi-coherent sheaf $_{\mathcal{P}}V$ on S by descent [FGI$^+$05, Theorem 4.2.3].

For a k-scheme Y with a left or right G-action and $\mathcal{P}$ a right G-bundle over S, we can form the *associated fiber bundle* $_{\mathcal{P}}Y$ over S and ask when it is representable by a scheme. This will be a key topic in the next subsection.

2.3. **Change of space.** Let $\beta : Z' \to Z$ be a G-equivariant morphism of schemes with right G-action. Then there is a natural morphism of stacks $[Z'/G] \to [Z/G]$ defined by sending

$$(\mathcal{P} \to Z') \mapsto (\mathcal{P} \to Z' \overset{\beta}{\to} Z).$$

The next lemma shows that under certain conditions, this morphism is schematic.

Lemma 2.3.1. *The morphism $[Z'/G] \to [Z/G]$ is representable. If the morphism of schemes $Z' \to Z$ is affine (resp. quasi-projective with a G-equivariant relatively ample invertible sheaf), then the morphism of quotient stacks is schematic and affine (resp. quasi-projective). If $Z' \to Z$ has a property that is fppf local on the target, then so does the morphism of quotient stacks.*

First, we need a formal result on 2-fibered products of quotient stacks.

Lemma 2.3.2. *Let $\beta_i : Z_i \to Z$ be G-equivariant morphisms of schemes with right G-action for $i = 1, 2$. Then the square*

$$
\begin{array}{ccc}
\left[(Z_1 \underset{Z}{\times} Z_2)/G\right] & \longrightarrow & [Z_1/G] \\
\downarrow & & \downarrow \\
[Z_2/G] & \longrightarrow & [Z/G]
\end{array}
$$

induced by the G-equivariant projections $Z_1 \times_Z Z_2 \to Z_i$ is 2-Cartesian.

Proof. For a scheme S, we have

$$\left([Z_1/G] \underset{[Z/G]}{\times} [Z_2/G]\right)(S) = \left\{ \begin{array}{c} f_1 : \mathcal{P}_1 \to Z_1, \ f_2 : \mathcal{P}_2 \to Z_2 \\ G\text{-equivariant morphism } \phi : \mathcal{P}_1 \to \mathcal{P}_2 \text{ over } S \times Z \end{array} \right\}.$$

Observe that we have a morphism from $(\mathcal{P}_1, f_1, \mathcal{P}_2, f_2, \phi)$ to $(\mathcal{P}_1, f_1, \mathcal{P}_1, f_2 \circ \phi, \mathrm{id}_{\mathcal{P}_1})$ via $\mathrm{id}_{\mathcal{P}_1}, \phi^{-1}$. The latter is the image of $(\mathcal{P}_1, (f_1, f_2 \circ \phi) : \mathcal{P}_1 \to Z_1 \times_Z Z_2) \in [(Z_1 \times_Z Z_2)/G](S)$. Hence the functor

$$F : \left[(Z_1 \underset{Z}{\times} Z_2)/G\right](S) \to \left([Z_1/G] \underset{[Z/G]}{\times} [Z_2/G]\right)(S)$$

is essentially surjective.

Take $(f : \mathcal{P} \to Z_1 \times_Z Z_2) \in [(Z_1 \times_Z Z_2)/G](S)$ and let $f_1 : \mathcal{P} \to Z_1, f_2 : \mathcal{P} \to Z_2$ be the morphisms corresponding to f. Then the image of $(\mathcal{P}, f)$ under F is $\tau = (\mathcal{P}, f_1, \mathcal{P}, f_2, \mathrm{id}_{\mathcal{P}})$. We similarly associate to $(\mathcal{P}', f') \in [(Z_1 \times_Z Z_2)/G](S)$ the corresponding f_1', f_2' and τ'. A morphism $\tau \to \tau'$ is then a pair of G-equivariant morphisms $\gamma_1 : \mathcal{P} \to \mathcal{P}'$ over $S \times Z_1$ and $\gamma_2 : \mathcal{P} \to \mathcal{P}'$ over $S \times Z_2$ intertwining $\mathrm{id}_{\mathcal{P}}$ and $\mathrm{id}_{\mathcal{P}'}$. The last condition implies $\gamma_1 = \gamma_2$. Thus γ_1 is a G-equivariant morphism over $S \times (Z_1 \times_Z Z_2)$, i.e., a morphism $(\mathcal{P}, f) \to (\mathcal{P}', f')$. We conclude that F is fully faithful and hence an isomorphism. $\qquad\square$

Proof of Lemma 2.3.1. For a scheme S, take $(f : \mathcal{P} \to Z) \in [Z/G](S)$. Remark 2.1.2 gives an isomorphism $S \simeq [\mathcal{P}/G]$. Hence by Lemma 2.3.2 we have a Cartesian square

$$
\begin{array}{ccc}
\left[(Z' \underset{Z}{\times} \mathcal{P})/G\right] & \longrightarrow & [Z'/G] \\
\downarrow & & \downarrow \\
S & \overset{\mathcal{P} \to Z}{\longrightarrow} & [Z/G]
\end{array}
$$

We prove that $[(Z' \times_Z \mathcal{P})/G]$ is representable by an algebraic space. Remark 2.1.2 implies that $[\mathcal{P}/G](T)$ is an equivalence relation for any scheme T. It follows that $[(Z' \times_Z \mathcal{P})/G](T)$ is an

equivalence relation, so $[(Z' \times_Z \mathcal{P})/G]$ is isomorphic to a sheaf of sets. Pick an fppf covering $(S_i \to S)$ so that each $\mathcal{P}_{S_i}$ admits a section $s_i \in \mathcal{P}(S_i)$. Note that we have a morphism $S_i \times G \to \mathcal{P}_{S_i} \to Z$. There is a G-equivariant isomorphism

$$\gamma_i : \left(Z' \underset{Z, f \circ s_i}{\times} S_i \right) \times G \to Z' \underset{Z}{\times} (S_i \times G)$$

over S_i where on the left hand side G acts as a trivial bundle over $Z' \times_{Z, f \circ s_i} S_i$, while on the right hand side G acts on Z' and $S_i \times G$. The morphism is defined by sending $(a, g) \mapsto (a.g, g)$ for $a \in Z'(T), g \in G(T)$ and T an S_i-scheme. Applying Remark 2.1.2 again, we have an isomorphism

$$\Phi_i : Z' \underset{Z, f \circ s_i}{\times} S_i \simeq \left[\left(Z' \underset{Z, f \circ s_i}{\times} S_i \right) \times G/G \right] \xrightarrow{\gamma_i} \left[\left(Z' \underset{Z}{\times} (S_i \times G) \right)/G \right] \xrightarrow{\mathrm{id} \times \widetilde{s_i}} \left[\left(Z' \underset{Z}{\times} \mathcal{P}_{S_i} \right)/G \right]$$

over S_i defined explicitly on T-points by sending $a \in (Z' \times_{Z, f \circ s_i} S_i)(T)$ to

$$T \times G \xrightarrow{a \times \mathrm{id}} \left(Z' \underset{Z, f \circ s_i}{\times} S_i \right) \times G \xrightarrow{\gamma_i} Z' \underset{Z}{\times} (S_i \times G) \xrightarrow{\mathrm{id} \times \widetilde{s_i}} Z' \underset{Z}{\times} \mathcal{P}$$

$$\downarrow$$

$$T$$

Therefore after restricting to an fppf covering, $[(Z' \times_Z \mathcal{P})/G] \times_S S_i \simeq [(Z' \times_Z \mathcal{P}_{S_i})/G]$ becomes representable by a scheme. Since algebraic spaces satisfy descent [Sta, Lemma 04SK], we deduce that $[(Z' \times_Z \mathcal{P})/G]$ is representable by an algebraic space. Before proving when $[(Z' \times_Z \mathcal{P})/G]$ is representable by a scheme, we give a description of its descent datum.

Descent datum of $[(Z' \times_Z \mathcal{P})/G]$. Let $S_{ij} = S_i \times_S S_j$. There are $g_{ij} \in G(S_{ij})$ such that $s_j = s_i.g_{ij} \in \mathcal{P}(S_{ij})$ (we slightly abuse notation and omit restriction signs when the context is clear). Then the action of g_{ij}^{-1} on Z' induces an isomorphism

$$\varphi_{ij} : Z' \underset{Z, f \circ s_j}{\times} S_{ij} \to Z' \underset{Z, (f \circ s_j).g_{ij}^{-1}}{\times} S_{ij} = Z' \underset{Z, f \circ s_i}{\times} S_{ij}.$$

It follows from the definitions that for $a \in (Z' \times_{Z, f \circ s_j} S_{ij})(T)$, the square

$$
\begin{array}{ccc}
T \times G & \xrightarrow{(\mathrm{id} \times \widetilde{s_j}) \circ \gamma_j \circ (a \times \mathrm{id})} & Z' \times_Z \mathcal{P}_{S_{ij}} \\
{\scriptstyle g_{ij}} \downarrow & & \| \\
T \times G & \xrightarrow{(\mathrm{id} \times \widetilde{s_i}) \circ \gamma_i \circ (\varphi_{ij} \circ (a \times \mathrm{id}))} & Z' \times_Z \mathcal{P}_{S_{ij}}
\end{array}
$$

is commutative. Therefore $\Phi_j(a)$ and $(\Phi_i \circ \varphi_{ij})(a)$ are isomorphic in $[(Z' \times_Z \mathcal{P}_{S_{ij}})/G]$. We conclude that $(Z' \times_{Z, f \circ s_i} S_i, \varphi_{ij})$ is a descent datum of $[(Z' \times_Z \mathcal{P})/G]$ with respect to the covering $(S_i \to S)$.

If $Z' \to Z$ is affine, then so is $Z' \times_{Z, f \circ s_i} S_i \to S_i$. Affine morphisms are effective under descent [FGI$^+$05, Theorem 4.33], so in this case $[(Z' \times_Z \mathcal{P})/G]$ is representable by a scheme affine over S. Next suppose that $Z' \to Z$ is quasi-projective with a G-equivariant relatively ample invertible $\mathcal{O}_{Z'}$-module $\mathcal{L}$. From our description of the descent datum of $[(Z' \times_Z \mathcal{P})/G]$,

we have a commutative square

$$
\begin{array}{ccc}
Z' \underset{Z, f \circ s_j}{\times} S_{ij} & \xrightarrow{\ \mathrm{pr}_{1,j}\ } & Z' \\
{\scriptstyle \varphi_{ij}}\big\downarrow & & \big\downarrow {\scriptstyle g_{ij}^{-1}} \\
Z' \underset{Z, f \circ s_i}{\times} S_{ij} & \xrightarrow{\ \mathrm{pr}_{1,i}\ } & Z'
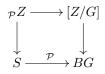\end{array}
$$

Therefore the G-equivariant structure of $\mathcal{L}$ gives an isomorphism

$$
\varphi_{ij}^* \, \mathrm{pr}_{1,i}^* \, \mathcal{L} \simeq \mathrm{pr}_{1,j}^* \, \mathcal{L}
$$

of invertible sheaves relatively ample over S_{ij}. Since the g_{ij} satisfy the cocycle condition, the associativity property of G-equivariance implies that the above isomorphisms satisfy the corresponding cocycle condition. We therefore have a descent datum of relatively ample invertible sheaves. By descent [SGA1, VIII, Proposition 7.8], we conclude that $[(Z' \times_Z \mathcal{P})/G]$ is representable by a scheme quasi-projective over S. $\qquad\square$

Corollary 2.3.3. *For a k-scheme Z with a right G-action and a right G-bundle $\mathcal{P}$ over a k-scheme S, there is an isomorphism $_{\mathcal{P}}Z \simeq [(Z \times \mathcal{P})/G]$ over S.*

Proof. The proof of Lemma 2.3.1 shows that $[(Z \times \mathcal{P})/G]$ has a descent datum $(Z \times S_i, \varphi_{ij})$. This is the same descent datum as $_{\mathcal{P}}Z$, proving the claim. This additionally implies that

$$
\begin{array}{ccc}
_{\mathcal{P}}Z & \longrightarrow & [Z/G] \\
\big\downarrow & & \big\downarrow \\
S & \xrightarrow{\ \mathcal{P}\ } & BG
\end{array}
$$

is a Cartesian square. $\qquad\square$

Corollary 2.3.4. *Given $\tau = (p : \mathcal{P} \to S, f : \mathcal{P} \to Z) \in [Z/G](S)$ for a k-scheme S,*

$$
\begin{array}{ccc}
\mathcal{P} & \xrightarrow{\ f\ } & Z \\
{\scriptstyle p}\big\downarrow & & \big\downarrow {\scriptstyle \tau_0} \\
S & \xrightarrow{\ \tau\ } & [Z/G]
\end{array}
$$

is a Cartesian square. In particular, $Z \to [Z/G]$ is schematic, affine, and fppf.

Proof. By Lemma 2.3.2, we have a Cartesian square

(2.3.4.1)
$$
\begin{array}{ccc}
\left[\big((Z \times G) \underset{\alpha, Z, f}{\times} \mathcal{P} \big)/G \right] & \longrightarrow & [Z \times G/G] \\
\big\downarrow & & \big\downarrow {\scriptstyle \alpha} \\
[\mathcal{P}/G] & \longrightarrow & [Z/G]
\end{array}
$$

where $Z \times G$ is the trivial G-bundle over Z. There is a G-equivariant morphism

$$
(f \times \mathrm{id}, \alpha_{\mathcal{P}}) : \mathcal{P} \times G \to (Z \times G) \underset{\alpha, Z, f}{\times} \mathcal{P}
$$

where $\mathcal{P} \times G$ is the trivial G-bundle over $\mathcal{P}$. Action and projection induce a G-equivariant morphism $\mathcal{P} \times G \to \mathcal{P} \times_S \mathcal{P}$ over $\mathcal{P} \times \mathcal{P}$. Therefore we see that the diagram

$$
\begin{array}{ccccc}
S & \xleftarrow{\ p\ } & \mathcal{P} & \xrightarrow{\ f\ } & Z \\
& & \downarrow{\scriptstyle \mathrm{id}:\mathcal{P} \times G \to \mathcal{P} \times G} & & \\
\mathrm{id}:\mathcal{P}\to\mathcal{P} \Big\downarrow & & [\mathcal{P} \times G/G] & & \Big\downarrow \mathrm{id}:Z \times G \to Z \times G \\
& & \downarrow{\scriptstyle (f \times \mathrm{id}, \alpha_{\mathcal{P}})} & & \\
[\mathcal{P}/G] & \longleftarrow & [((Z \times G) \underset{\alpha,Z,f}{\times} \mathcal{P})/G] & \longrightarrow & [Z \times G/G]
\end{array}
$$

is 2-commutative. Applying Remark 2.1.2 multiple times, we deduce that (2.3.4.1) is isomorphic to the desired Cartesian square of the claim. Moreover, we see that the morphism $f^*\tau_0 \to p^*\tau$ is defined by the action and first projection morphisms. $\qquad\square$

Remark 2.3.5. Let Z be a k-scheme and give it the trivial G-action. Lemma 2.1.6 implies that $\cdot \to BG$ is G-invariant, so by base change $Z \to BG \times Z$ is also G-invariant. For any morphism $(\mathcal{P}, u) : S \to BG \times Z$, the fibered product $Z \times_{BG \times Z} S \simeq \cdot \times_{BG} S$ is isomorphic to $\mathcal{P}$ by Corollary 2.3.4, and the morphism $\mathcal{P} \to Z$ is $\mathcal{P} \to S$ composed with u. Lemma 2.1.1 then gives an isomorphism $BG \times Z \simeq [Z/G]$. This isomorphism implies that for a G-bundle $\mathcal{P}$ on S, any G-invariant morphism $\mathcal{P} \to Z$ factors through a unique morphism $S \to Z$.

2.4. **Change of group.** Let $H \hookrightarrow G$ be a closed subgroup of G. We want to show the following relation between BH and BG.

Lemma 2.4.1. *The morphism* $BH \to BG$ *sending* $\mathcal{P} \mapsto {}_{\mathcal{P}}G$ *is schematic, finitely presented, and quasi-projective.*

Here we are twisting G by the left H-action. As we shall see, $\mathcal{P} \mapsto {}_{\mathcal{P}}G$ gives a well-defined morphism of stacks $BH \to BG$.
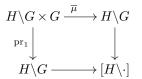
By [DG70, III, §3, Théorème 5.4], the fppf sheaf $H\backslash G$ of left cosets is representable by a quasi-projective k-scheme with a G-equivariant, ample invertible sheaf. The morphism $H \times G \to G \times_{H\backslash G} G$ induced by multiplication and projection is an isomorphism by [DG70, III, §1, 2.4]. Therefore the projection $\pi : G \to H\backslash G$ is a left H-bundle. Note that there is an obvious analogue of our previous discussion of quotient stacks for schemes with left actions instead of right actions. We use $[H\backslash G]$ to denote the stack quotient by the left action. Then Remark 2.1.2 implies that $\mathrm{id} : G \to G$ induces an isomorphism $H\backslash G \to [H\backslash G]$.

Lemma 2.4.2. *The left H-bundle* $G \to H\backslash G$ *defines a right G-bundle* $H\backslash G \to [H\backslash\cdot]$.
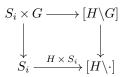
Proof. Let $\mu : G \times G \to G$ denote the group multiplication. We have an H-equivariant morphism $(\mathrm{pr}_1, \mu) : G \times G \to G \times G$ where H acts on the first coordinate on the left hand side, and H acts diagonally on the right hand side. We have 2-commutative squares

$$
\begin{array}{ccccc}
H\backslash G & \xleftarrow{\ \mathrm{pr}_1\ } & H\backslash G \times G & \xrightarrow{\ \overline{\mu}\ } & H\backslash G \\
\mathrm{id}:G\to G \Big\downarrow & & \Big\downarrow{\scriptstyle (\mathrm{pr}_1,\mu):G \times G \to G \times G} & & \Big\downarrow \mathrm{id}:G\to G \\
[H\backslash G] & \xleftarrow{\ \mathrm{pr}_1\ } & [H\backslash(G \times G)] & \xrightarrow{\ \mathrm{pr}_2\ } & [H\backslash G]
\end{array}
$$

where the 2-morphisms are just $\mathrm{id}_{G \times G}$. Thus we have a 2-commutative square

$$
\begin{array}{ccc}
H\backslash G \times G & \xrightarrow{\overline{\mu}} & H\backslash G \\
{\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow \\
H\backslash G & \longrightarrow & [H\backslash \cdot]
\end{array}
$$

with 2-morphism $\mathrm{id}_{G \times G}$, which gives $H\backslash G \to [H\backslash \cdot]$ the structure of a right G-invariant morphism. Take a left H-bundle $\mathcal{P} \in [H\backslash \cdot](S)$ and an fppf covering $(S_i \to S)$ trivializing $\mathcal{P}$. From the proof of Lemma 2.3.1, we find that

$$
\begin{array}{ccc}
S_i \times G & \longrightarrow & [H\backslash G] \\
\downarrow & & \downarrow \\
S_i & \xrightarrow{H \times S_i} & [H\backslash \cdot]
\end{array}
$$

is Cartesian, the top arrow corresponds to $H \times S_i \times G \to G : (h, a, g) \mapsto hg$, and the associated 2-morphism is $\mathrm{id}_{H \times S_i \times G}$. Observe that the morphism $S_i \times G \to [H\backslash G] \simeq H\backslash G$ equals $\pi \circ \mathrm{pr}_2$, which is G-equivariant. Therefore $S_i \times G \to H\backslash G \times_{[H\backslash \cdot], H \times S_i} S_i \simeq (H\backslash G \times_{[H\backslash \cdot], \mathcal{P}} S) \times_S S_i$ is a G-equivariant isomorphism. Thus $H\backslash G \times_{[H\backslash \cdot], \mathcal{P}} S$ is a right G-bundle, which proves that $H\backslash G \to [H\backslash \cdot]$ is also a right G-bundle. $\qquad\square$

Now Lemmas 2.1.1 and 2.4.2 give an isomorphism $[H\backslash \cdot] \simeq [(H\backslash G)/G]$ sending a left H-bundle $\mathcal{P}$ over S to $H\backslash G \times_{[H\backslash \cdot], \mathcal{P}} S \to H\backslash G$. Left H-bundles are equivalent to right H-bundles using the inverse action, so $BH \simeq [H\backslash \cdot]$. By keeping track of left and right actions and applying Corollary 2.3.3, we observe that the morphism $BH \simeq [H\backslash \cdot] \simeq [(H\backslash G)/G] \to BG$ sends a right H-bundle $\mathcal{P}$ to the associated fiber bundle $_\mathcal{P}G$, where G is twisted by the left H-action. The right G-action on $_\mathcal{P}G$ is induced by multiplication on the right by G.

Proof of Lemma 2.4.1. Applying Corollary 2.3.3 to the morphism $[(H\backslash G)/G] \to BG$, we get a 2-Cartesian square

$$
\begin{array}{ccc}
_\mathcal{E}(H\backslash G) & \longrightarrow & BH \\
\downarrow & & \downarrow \\
S & \xrightarrow{\mathcal{E}} & BG
\end{array}
$$

Since $H\backslash G$ is quasi-projective with a G-equivariant ample invertible sheaf over $\mathrm{Spec}\, k$, we have that $_\mathcal{E}(H\backslash G)$ is representable by a scheme quasi-projective and of finite presentation over S by Lemma 2.3.1. $\qquad\square$

2.5. **Proving $[Z/G]$ is algebraic.** We now prove Theorem 2.0.2.

Lemma 2.5.1. *For algebraic groups G, G', the morphism $BG \times BG' \to B(G \times G')$ sending a G-bundle $\mathcal{P}$ and a G'-bundle $\mathcal{P}'$ over S to $\mathcal{P} \times_S \mathcal{P}'$ is an isomorphism.*

Proof. Take $(\mathcal{P}, \mathcal{P}') \in (BG \times BG')(S)$ and choose an fppf covering $(S_i \to S)$ on which both $\mathcal{P}$ and $\mathcal{P}'$ are trivial. Then we have descent data

$$
(S_i \times G, g_{ij}), \ (S_i \times G', g'_{ij})
$$

for $g_{ij} \in G(S_i \times_S S_j), g'_{ij} \in G'(S_i \times_S S_j)$, corresponding to $\mathcal{P}, \mathcal{P}'$, respectively. Then
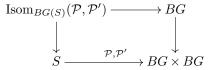
$$
(S_i \times G \times G', (g_{ij}, g'_{ij}))
$$

gives a descent datum for $\mathcal{P} \times_S \mathcal{P}'$. Conversely, any such descent datum with (g_{ij}, g'_{ij}) satisfying the cocycle condition implies that g_{ij} and g'_{ij} satisfy the cocycle condition separately. Since $BG \times BG'$ and $B(G \times G')$ are both stacks, we deduce that $(\mathcal{P}, \mathcal{P}') \mapsto \mathcal{P} \times_S \mathcal{P}'$ gives an isomorphism. From the descent data, we see that an inverse to this morphism is the morphism sending $\mathcal{E} \mapsto (_{\mathcal{E}}G, _{\mathcal{E}}G')$, where $G \times G'$ acts by the projections on G, G'. $\qquad \square$

Lemma 2.5.2. *The k-stack BG is an algebraic stack with a schematic, affine diagonal. Specifically, for right G-bundles $\mathcal{P}, \mathcal{P}'$ over S, there is an isomorphism*

$$\mathrm{Isom}_{BG(S)}(\mathcal{P}, \mathcal{P}') \simeq {}_{\mathcal{P} \times_S \mathcal{P}'}(G)$$

as sheaves of sets on $\mathbf{Sch}_{/S}$, where $G \times G$ acts on G from the right by $g.(g_1, g_2) = g_1^{-1} g g_2$.

Proof. For a scheme S and $\mathcal{P}, \mathcal{P}' \in BG(S)$, we have a Cartesian square

$$
\begin{array}{ccc}
\mathrm{Isom}_{BG(S)}(\mathcal{P}, \mathcal{P}') & \longrightarrow & BG \\
\downarrow & & \downarrow \\
S & \xrightarrow{\ \mathcal{P}, \mathcal{P}'\ } & BG \times BG
\end{array}
$$

Observe that there is an isomorphism $G \backslash (G \times G) \simeq G$ by sending $(g_1, g_2) \mapsto g_1^{-1} g_2$. The induced right action of $G \times G$ on G is then $g.(g_1, g_2) = g_1^{-1} g g_2$ as claimed. By the proofs of Lemmas 2.4.1 and 2.5.1, we get an isomorphism $\mathrm{Isom}_{BG(S)}(\mathcal{P}, \mathcal{P}') \simeq {}_{\mathcal{P} \times_S \mathcal{P}'}(G)$ over S. Since G is affine over k, the associated fiber bundle ${}_{\mathcal{P} \times_S \mathcal{P}'}(G)$ is representable by a scheme affine over S. Hence the diagonal $BG \to BG \times BG$ is schematic and affine. As a special case of Lemma 2.3.4, the morphism $\cdot \to BG$ is schematic, affine, and fppf. By Artin's Theorem [LMB00, Théorème 10.1], BG is an algebraic stack (if G is smooth over k, the last step is unnecessary). $\qquad \square$

Proof of Theorem 2.0.2. Fix a scheme S and $(\mathcal{P} \to Z) \in [Z/G](S)$. Then by Remark 2.1.2 and Lemma 2.3.1, the morphism $S \simeq [\mathcal{P}/G] \to [Z/G]$ is representable. Therefore the diagonal $\Delta_{[Z/G]}$ is representable. By Lemma 2.5.2, there exists a scheme U and a smooth surjective morphism $U \to BG$. The change of space morphism $f : [Z/G] \to BG$ is representable by Lemma 2.3.1, so $U \times_{BG} [Z/G]$ is representable by an algebraic space. Therefore there exists a scheme V with an étale surjective morphism $V \to U \times_{BG} [Z/G]$. The composition $V \to [Z/G]$ thus gives a presentation. This shows that $[Z/G]$ is an algebraic stack.

The diagonal $\Delta_{[Z/G]}$ is the composition of the morphisms

$$[Z/G] \xrightarrow{\Delta_f} [Z/G] \underset{BG}{\times} [Z/G] \to [Z/G] \times [Z/G].$$

We have a 2-Cartesian square

$$
\begin{array}{ccc}
[Z/G] \underset{BG}{\times} [Z/G] & \longrightarrow & [Z/G] \times [Z/G] \\
\downarrow & & \downarrow \\
BG & \longrightarrow & BG \times BG
\end{array}
$$

and we know the diagonal Δ_{BG} is schematic and affine by Lemma 2.5.2. By base change, this implies $[Z/G] \times_{BG} [Z/G] \to [Z/G] \times [Z/G]$ is also schematic and affine. Since f is representable, [Sta, Lemma 04YQ] implies that Δ_f is schematic and separated. If Z is quasi-separated, then by descent ${}_{\mathcal{P}}Z \to S$ is quasi-separated for any $\mathcal{P} \in BG(S)$. Thus Corollary 2.3.3 implies that f is quasi-separated, and [Sta, Lemma 04YT] shows that Δ_f is quasi-compact. If Z is separated, we deduce by the same reasoning that f is separated. Hence by [Sta, Lemma 04YS],

the relative diagonal Δ_f is a closed immersion. Composition of the two morphisms Δ_f and $[Z/G] \times_{BG} [Z/G] \to [Z/G] \times [Z/G]$ now gives the claimed properties of $\Delta_{[Z/G]}$. $\square$

3. Hom stacks

For a base scheme S, let X be an S-scheme and $\mathcal{Y} : (\mathbf{Sch}_{/S})^{\mathrm{op}} \to \mathbf{Gpd}$ a pseudo-functor. Then we define the Hom 2-functor $\mathcal{H}om_S(X, \mathcal{Y})$ by

$$\mathcal{H}om_S(X, \mathcal{Y})(T) = \mathrm{Hom}_T(X_T, \mathcal{Y}_T) = \mathrm{Hom}_S(X_T, \mathcal{Y}),$$

sending an S-scheme T to the 2-category of groupoids. By the 2-Yoneda lemma, we have a natural equivalence of categories $\mathrm{Hom}_S(X_T, \mathcal{Y}) \simeq \mathcal{Y}(X_T)$. From this we deduce that if $\mathcal{Y}$ is an fpqc S-stack, then $\mathcal{H}om_S(X, \mathcal{Y})$ is as well.

The main example of a Hom stack we will be concerned with is Bun_G:

Definition 3.0.3. Let S be a k-scheme. Then for an S-scheme X, we define the moduli stack of G-bundles on $X \to S$ by $\mathrm{Bun}_G = \mathcal{H}om_S(X, BG \times S)$.

As we remarked earlier, Bun_G is an fpqc stack since BG is. Explicitly, for an S-scheme T, $\mathrm{Bun}_G(T)$ is the groupoid of right G-bundles on X_T.

In this section, we prove some properties on morphisms between Hom stacks. As a corollary, we show that the diagonal of Bun_G is schematic and affine under certain conditions. These properties reduce to understanding the sheaf of sections associated to a morphism of schemes, which we now introduce.

3.1. **Scheme of sections.** Let S be a base scheme and $X \to S$ a morphism of schemes. Given another morphism of schemes $Y \to X$, we define the presheaf of sets $\mathrm{Sect}_S(X, Y)$ on $\mathbf{Sch}_{/S}$ to send

$$(T \to S) \mapsto \mathrm{Hom}_{X_T}(X_T, Y_T) = \mathrm{Hom}_X(X_T, Y).$$

Since schemes represent fpqc sheaves, the presheaf $\mathrm{Sect}_S(X, Y)$ is an fpqc sheaf of sets. Here is the main result of this subsection:

Theorem 3.1.1. *Let $X \to S$ be a flat, finitely presented, projective morphism, and let $Y \to X$ be a finitely presented, quasi-projective morphism. Then $\mathrm{Sect}_S(X, Y)$ is representable by a disjoint union of schemes which are finitely presented and locally quasi-projective over S.*

Example 3.1.2. Observe that if X and Z are schemes over S, then taking the first projection $X \times_S Z \to X$ gives an equality $\mathrm{Sect}_S(X, X \times_S Z) = \mathcal{H}om_S(X, Z)$.

We first consider when $Y \to X$ is affine, which will be of independent interest because we get a stronger result. We then prove the theorem by considering the cases where $Y \to X$ is an open immersion and a projective morphism separately.

Lemma 3.1.3. *Let $p : X \to S$ be a flat, finitely presented, proper morphism and $\mathcal{E}$ a locally free $\mathcal{O}_X$-module of finite rank. Then $\mathrm{Sect}_S(X, \mathbf{Spec}_X \mathrm{Sym}_{\mathcal{O}_X} \mathcal{E})$ is representable by a scheme affine and finitely presented over S.*

Proof. By [EGAG, 0, Corollaire 4.5.5], we can take an open covering of S and reduce to considering the case where S is affine. Since any ring is an inductive limit of finitely presented $\mathbf{Z}$-algebras, by [EGAIV3, §8] there exist an affine scheme S_1 of finite type over $\mathrm{Spec}\,\mathbf{Z}$ with a morphism $S \to S_1$, a flat proper morphism $X_1 \to S_1$, and a locally free $\mathcal{O}_{X_1}$-module $\mathcal{E}_1$ such that $X_1, \mathcal{E}_1$ base change to $X, \mathcal{E}$ under $S \to S_1$. Therefore

$$\mathrm{Sect}_S(X, \mathbf{Spec}_X \mathrm{Sym}_{\mathcal{O}_X} \mathcal{E}) \simeq \mathrm{Sect}_{S_1}(X_1, \mathbf{Spec}_{X_1} \mathrm{Sym}_{\mathcal{O}_{X_1}} \mathcal{E}_1) \underset{S_1}{\times} S$$

over S. Replacing S by S_1, we can assume that S is affine and noetherian.

Let $\mathcal{F} = \mathcal{E}^\vee$, which is a locally free $\mathcal{O}_X$-module. From [EGAIII2, Théorème 7.7.6, Remarques 7.7.9], there exists a coherent $\mathcal{O}_S$-module $\mathcal{Q}$ equipped with natural isomorphisms

$$(3.1.3.1) \qquad\qquad \operatorname{Hom}_{\mathcal{O}_T}(\mathcal{Q}_T, \mathcal{O}_T) \simeq \Gamma(T, p_{T*}\mathcal{F}_T) = \Gamma(X_T, \mathcal{F}_T)$$
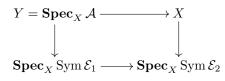
for any S-scheme T. A section $X_T \to (\mathbf{Spec}_X \operatorname{Sym} \mathcal{E}) \times_S T$ over X_T is equivalent to a morphism of $\mathcal{O}_{X_T}$-modules $\mathcal{E}_T \to \mathcal{O}_{X_T}$. The latter is by definition a global section of $\Gamma(X_T, \mathcal{E}_T^\vee)$. Since $\mathcal{E}$ is locally free, we have a canonical isomorphism $\mathcal{F}_T \simeq \mathcal{E}_T^\vee$. The isomorphism of $(3.1.3.1)$ therefore shows that an element of $\operatorname{Sect}_S(X, \mathbf{Spec}_X \operatorname{Sym} \mathcal{E})(T)$ is naturally isomorphic to a morphism in $\operatorname{Hom}_S(T, \mathbf{Spec}_S \operatorname{Sym}_{\mathcal{O}_S} \mathcal{Q}) \simeq \operatorname{Hom}_{\mathcal{O}_T}(\mathcal{Q}_T, \mathcal{O}_T)$. We conclude that

$$\operatorname{Sect}_S(X, \mathbf{Spec}_X \operatorname{Sym}_{\mathcal{O}_X} \mathcal{E}) \simeq \mathbf{Spec}_S \operatorname{Sym}_{\mathcal{O}_S} \mathcal{Q}$$

as sheaves over S. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 3.1.4. *Let $p : X \to S$ be a flat, finitely presented, projective morphism. If $Y \to X$ is affine and finitely presented, then $\operatorname{Sect}_S(X, Y)$ is representable by a scheme affine and finitely presented over S.*

Proof. We make the same reductions as in the proof of Lemma 3.1.3 to assume S is affine and noetherian, and X is a closed subscheme of $\mathbf{P}_S^r$ for some integer r. Since Y is affine over X, it has the form $Y = \mathbf{Spec}_X \mathcal{A}$ for a quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{A}$. By finite presentation of Y over X and quasi-compactness of X, there are finitely many local sections of $\mathcal{A}$ that generate $\mathcal{A}$ as an $\mathcal{O}_X$-algebra. By extending coherent sheaves [EGAI, Corollaire 9.4.3], there exists a coherent $\mathcal{O}_X$-module $\mathcal{F} \subset \mathcal{A}$ that contains all these sections. From [Har77, II, Corollary 5.18], there exists a locally free sheaf $\mathcal{E}_1$ on X surjecting onto $\mathcal{F}$. This induces a surjection of $\mathcal{O}_X$-algebras $\operatorname{Sym}_{\mathcal{O}_X} \mathcal{E}_1 \twoheadrightarrow \mathcal{A}$. Let $\mathcal{I} \subset \operatorname{Sym} \mathcal{E}_1$ be the ideal sheaf. Apply the same argument again to get a locally free sheaf $\mathcal{E}_2$ on X and a morphism $\mathcal{E}_2 \to \mathcal{I}$ whose image generates $\mathcal{I}$ as a $\operatorname{Sym}_{\mathcal{O}_X} \mathcal{E}_1$-module. Therefore we have a Cartesian square

$$
\begin{array}{ccc}
Y = \mathbf{Spec}_X \mathcal{A} & \longrightarrow & X \\
\downarrow & & \downarrow \\
\mathbf{Spec}_X \operatorname{Sym} \mathcal{E}_1 & \longrightarrow & \mathbf{Spec}_X \operatorname{Sym} \mathcal{E}_2
\end{array}
$$

over X, where $X \to \mathbf{Spec}_X \operatorname{Sym} \mathcal{E}_2$ is the zero section. By the universal property of the fibered product, we have a canonical isomorphism

$$\operatorname{Sect}_S(X, Y) \simeq \operatorname{Sect}_S(X, X) \underset{\operatorname{Sect}_S(X, \mathbf{Spec}_X \operatorname{Sym} \mathcal{E}_2)}{\times} \operatorname{Sect}_S(X, \mathbf{Spec}_X \operatorname{Sym} \mathcal{E}_1).$$

Note that $S \simeq \operatorname{Sect}_S(X, X)$. By Lemma 3.1.3, all three sheaves in the fibered product are representable by schemes affine and finitely presented over S. Therefore $\operatorname{Sect}_S(X, Y)$ is representable by a scheme. The morphism $S \simeq \operatorname{Sect}_S(X, X) \to \operatorname{Sect}_S(X, \mathbf{Spec}_X \operatorname{Sym} \mathcal{E}_2)$ sends $T \to S$ to $0 \in \Gamma(X_T, (\mathcal{E}_2)_T^\vee)$, using the correspondence in Lemma 3.1.3. Therefore

$$S \to \operatorname{Sect}_S(X, \mathbf{Spec}_X \operatorname{Sym} \mathcal{E}_2) \simeq \mathbf{Spec}_S \operatorname{Sym} \mathcal{Q}_2$$

is the zero section, where $\mathcal{Q}_2$ is as in Lemma 3.1.3. In particular, it is a closed immersion. By base change, $\operatorname{Sect}_S(X, Y) \to \operatorname{Sect}_S(X, \mathbf{Spec}_X \operatorname{Sym} \mathcal{E}_1)$ is also a closed immersion. Therefore $\operatorname{Sect}_S(X, Y)$ is affine and finitely presented over S. $\qquad\qquad\qquad\square$

Remark 3.1.5. If $Y \to X$ is a finitely presented closed immersion, then note that we can take $\mathcal{E}_1 = 0$ in the proof of Lemma 3.1.4. It follows that $\operatorname{Sect}_S(X, Y) \to S$ is a closed immersion.

Lemma 3.1.6. *Let $p : X \to S$ be a proper morphism. For any morphism $Y \to X$ and $U \hookrightarrow Y$ an open immersion, $\mathrm{Sect}_S(X, U) \to \mathrm{Sect}_S(X, Y)$ is schematic and open.*

Proof. For an S-scheme T, suppose we have a morphism $f : X_T \to Y$ over S corresponding to a morphism of sheaves $T \to \mathrm{Sect}_S(X, Y)$ by the Yoneda lemma. Then the fibered product

$$T \underset{\mathrm{Sect}_S(X,Y)}{\times} \mathrm{Sect}_S(X, U)$$

is a sheaf on $\mathbf{Sch}_{/T}$ sending $T' \to T$ to a singleton set if $X_{T'} \to X_T \to Y$ factors through U and to the empty set otherwise. We claim that this sheaf is representable by the open subscheme

$$T - p_T(X_T - f^{-1}(U))$$

of T, where $p_T(X_T - f^{-1}(U))$ is a closed subset of T by properness of p_T. Suppose that the image of $T' \to T$ contains a point of $p_T(X_T - f^{-1}(U))$. In other words, there are $t' \in T'$ and $x \in X_T$ mapping to the same point of T, and $f(x) \notin U$. Then the compositum field of $\kappa(t'), \kappa(x)$ over $\kappa(p_T(x))$ gives a point of $X_{T'}$ mapping to t', x. This shows that $X_{T'} \to Y$ does not factor through U. For the converse, suppose that there is a point $x' \in X_{T'}$ with image outside of U. Then the image $x \in X_T$ is not in $f^{-1}(U)$, so $p_{T'}(x')$ maps to $p_T(x) \in p_T(X_T - f^{-1}(U))$. We conclude that $T' \to T$ factors through $T - p_T(X_T - f^{-1}(U))$ if and only if $X_{T'} \to Y$ factors through U. Therefore $T \times_{\mathrm{Sect}_S(X,Y)} \mathrm{Sect}_S(X, U)$ is representable by this open subscheme of T. $\qquad\square$

Lemma 3.1.7. *Let S be a noetherian scheme, and let $p : X \to S$ and $Z \to S$ be flat proper morphisms. Suppose there is a morphism $\pi : Z \to X$ over S. Then there exists an open subscheme $S_1 \subset S$ with the following universal property. For any locally noetherian S-scheme T, the base change $\pi_T : Z_T \to X_T$ is an isomorphism if and only if $T \to S$ factors through S_1.*

Proof. Since X and Z are both proper over S, we deduce that π is a proper morphism. By [FGI$^+$05, Theorem 5.22(a)], we can assume that π is flat. From Chevalley's upper semi-continuity theorem for dimension of fibers [EGAIV3, Corollaire 13.1.5], the set of $x \in X$ such that $\pi^{-1}(x)$ is finite form an open subset $U_1 \subset X$. The restriction $\pi : \pi^{-1}(U_1) \to U_1$ is a flat, finite morphism by [EGAIII1, Proposition 4.4.2]. Therefore $\pi_* \mathcal{O}_{\pi^{-1}(U_1)}$ is a locally free $\mathcal{O}_{U_1}$-module. Let U be the set of $x \in U_1$ where $\mathcal{O}_{U_1} \otimes \kappa(x) \to \pi_* \mathcal{O}_{\pi^{-1}(U_1)} \otimes \kappa(x)$ is an isomorphism. Since $\mathcal{O}_{U_1} \to \pi_* \mathcal{O}_{\pi^{-1}(U_1)}$ is a morphism between locally free $\mathcal{O}_{U_1}$-modules, U is the maximal open subscheme of X on which this morphism is an isomorphism, which is equivalent to being the maximal open such that $\pi^{-1}(U) \to U$ is an isomorphism. Since the property of being an isomorphism is fpqc local on the base, the same argument as in the proof of Lemma 3.1.6 shows that $S_1 = S - p(X - U)$ is the open subscheme with the desired universal property. $\qquad\square$

Suppose we have a proper morphism $X \to S$ and a separated morphism $Y \to X$. Then for a section $f \in \mathrm{Sect}_S(X, Y)(T)$, the graph of f over X_T is a closed immersion $X_T \to X_T \times_{X_T} Y_T$. The isomorphism $X_T \times_{X_T} Y_T \to Y_T$ implies f is a closed immersion. Therefore $f : X_T \hookrightarrow Y_T$ represents an element of $\mathrm{Hilb}_{Y/S}(T)$. Therefore we have defined an injection of sheaves

$$(3.1.7.1) \qquad\qquad\qquad \mathrm{Sect}_S(X, Y) \to \mathrm{Hilb}_{Y/S}.$$

Lemma 3.1.8. *Let $p : X \to S$ and $Y \to S$ be finitely presented, proper morphisms, and suppose that p is flat. Then (3.1.7.1) is an open immersion.*

Proof. The assertion is Zariski local on the base, so we can use [EGAIV3, §8] as in the proof of Lemma 3.1.3 to assume S is affine and noetherian. Let $T \to \mathrm{Hilb}_{Y/S}$ represent $Z \subset Y_T$, a closed subscheme flat over an affine S-scheme T. We may assume T is noetherian by [EGAIV3, §8]. Applying Lemma 3.1.7 to the composition $Z \to X_T$, we deduce that there exists an

open subscheme $U \subset T$ such that for any locally noetherian scheme $T' \to T$, the base change $Z_{T'} \to X_{T'}$ is an isomorphism if and only if $T' \to T$ factors through U. Observe that an isomorphism $Z_{T'} \to X_{T'}$ corresponds uniquely to a section $X_{T'} \simeq Z_{T'} \hookrightarrow Y_{T'}$. The locally noetherian hypothesis on T' can be removed as usual. Therefore $T \times_{\mathrm{Hilb}_{Y/S}} \mathrm{Sect}_S(X, Y)$ is represented by U. $\qquad\square$

Proof of Theorem 3.1.1. Let $\mathcal{L}$ be an invertible $\mathcal{O}_Y$-module ample relative to $\pi : Y \to X$ and $\mathcal{K}$ an invertible $\mathcal{O}_X$-module ample relative to $p : X \to S$. Choose an integer $\chi_{n,m}$ for every pair of integers n, m. Define the subfunctor

$$\mathrm{Hilb}_{Y/S}^{(\chi_{n,m})} \subset \mathrm{Hilb}_{Y/S}$$

to have T-points the closed subschemes $Z \in \mathrm{Hilb}_{Y/S}(T)$ such that for all $t \in T$, the Euler characteristic $\chi((\mathcal{O}_Z \otimes \mathcal{L}^{\otimes n} \otimes \pi^*(\mathcal{K}^{\otimes m}))_t) = \chi_{n,m}$. We claim that the $\mathrm{Hilb}_{Y/S}^{(\chi_{n,m})}$ form a disjoint open cover of $\mathrm{Hilb}_{Y/S}$. This can be checked Zariski locally on S, so we can assume S is noetherian using [EGAIV3, §8]. Then $\mathrm{Hilb}_{Y/S}$ is representable by a locally noetherian scheme [AK80, Corollary 2.7]. Since the Euler characteristic is locally constant [EGAIII2, Théorème 7.9.4] and the connected components of a locally noetherian scheme are open [EGAI, Corollaire 6.1.9], we deduce the claim.

Now it suffices to show that for any choice of $(\chi_{n,m})$, the functor

$$\mathrm{Sect}_S(X, Y) \cap \mathrm{Hilb}_{Y/S}^{(\chi_{n,m})}$$

is representable by a scheme finitely presented and locally quasi-projective over S. The assertion is Zariski local on S [EGAG, 0, Corollaire 4.5.5], so we can assume that S is affine and noetherian. Now by [EGAII, Propositions 4.4.6, 4.6.12, 4.6.13], there exists a scheme $\overline{Y}$ projective over X, an open immersion $Y \hookrightarrow \overline{Y}$, an invertible module $\overline{\mathcal{L}}$ very ample relative to $\overline{Y} \to S$, and positive integers a, b such that

$$\mathcal{L}^{\otimes a} \otimes \pi^*(\mathcal{K}^{\otimes b}) \simeq \overline{\mathcal{L}}|_Y.$$

Let $\Phi \in \mathbf{Q}[\lambda]$ be a polynomial such that $\Phi(n) = \chi_{na,nb}$ for all integers n (if no such polynomial exists, then $\mathrm{Hilb}_{Y/S}^{(\chi_{n,m})}$ is empty). Lemmas 3.1.6 and 3.1.8 and [AK80, Theorem 2.6, Step IV] imply that $\mathrm{Sect}_S(X, Y) \cap \mathrm{Hilb}_{Y/S}^{(\chi_{n,m})}$ is an open subfunctor of $\mathrm{Hilb}_{\overline{Y}/S}^{\Phi, \overline{\mathcal{L}}}$. The claim now follows from [AK80, Corollary 2.8] as an open immersion to a noetherian scheme is finitely presented. $\qquad\square$

3.2. **Morphisms between Hom stacks.** The goal of this subsection is to use our results on the scheme of sections to deduce that the diagonal of Bun_G is schematic when X is flat, finitely presented, and projective over S.

Lemma 3.2.1. *Suppose that $X \to S$ is flat, finitely presented, and projective. Let $F : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a schematic morphism between pseudo-functors. If F is quasi-projective (resp. affine) and of finite presentation, then the corresponding morphism*

$$\mathcal{H}om_S(X, \mathcal{Y}_1) \to \mathcal{H}om_S(X, \mathcal{Y}_2)$$

is schematic and locally of finite presentation (resp. affine and of finite presentation).

Before proving the lemma, we mention the corollaries of interest.

Corollary 3.2.2. *Suppose that $X \to S$ is flat, finitely presented, and projective. Then the diagonal of Bun_G is schematic, affine, and finitely presented.*

Proof. By Lemma 2.5.1, we deduce that the canonical morphism $\mathrm{Bun}_{G \times G} \to \mathrm{Bun}_G \times \mathrm{Bun}_G$ is an isomorphism. Applying Lemmas 2.5.2 and 3.2.1 to $BG \to B(G \times G)$, we deduce that $\mathrm{Bun}_G \to \mathrm{Bun}_{G \times G}$ is schematic, affine, and finitely presented, which proves the claim. $\qquad\square$

Remark 3.2.3. In line with Remark 2.0.3, we note that Corollary 3.2.2 holds if G is a group scheme affine and of finite presentation over a base S.

Corollary 3.2.4. *Let $H \hookrightarrow G$ be a closed subgroup of G. If $X \to S$ is flat, finitely presented, and projective, then the corresponding morphism $\mathrm{Bun}_H \to \mathrm{Bun}_G$ is schematic and locally of finite presentation.*

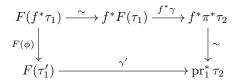*Proof.* This follows from Lemmas 2.4.1 and 3.2.1. $\qquad\square$

Proof of Lemma 3.2.1. Let $\tau_2 : X_T \to \mathcal{Y}_2$ represent a morphism from $T \to \mathcal{H}om_S(X, \mathcal{Y}_2)$. Since F is schematic, the 2-fibered product $\mathcal{Y}_1 \times_{\mathcal{Y}_2, \tau_2} X_T$ is representable by a scheme Y_T. Let

$$
\begin{array}{ccc}
Y_T & \xrightarrow{\ \tau_1\ } & \mathcal{Y}_1 \\
{\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle F} \\
X_T & \xrightarrow{\ \tau_2\ } & \mathcal{Y}_2
\end{array}
$$

be the 2-Cartesian square, with a 2-morphism $\gamma : F(\tau_1) \simeq \pi^* \tau_2$. Now for a T-scheme T', suppose that $\tau_1' : X_{T'} \to \mathcal{Y}_1$ is a 1-morphism such that the square

$$
\begin{array}{ccc}
X_{T'} & \xrightarrow{\ \tau_1'\ } & \mathcal{Y}_1 \\
{\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle F} \\
X_T & \xrightarrow{\ \tau_2\ } & \mathcal{Y}_2
\end{array}
$$

is 2-commutative via a 2-morphism $\gamma' : F(\tau_1') \simeq \mathrm{pr}_1^* \tau_2$. By the definition of 2-fibered products and the assumptions on Y_T, there exists a unique morphism of schemes $f : X_{T'} \to Y_T$ over X_T and a unique 2-morphism $\phi : f^* \tau_1 \simeq \tau_1'$ such that

$$
\begin{array}{ccc}
F(f^*\tau_1) & \xrightarrow{\ \sim\ } f^*F(\tau_1) \xrightarrow{\ f^*\gamma\ } & f^*\pi^*\tau_2 \\
{\scriptstyle F(\phi)}\downarrow & & \downarrow{\scriptstyle \sim} \\
F(\tau_1') & \xrightarrow{\quad\quad\gamma'\quad\quad} & \mathrm{pr}_1^* \tau_2
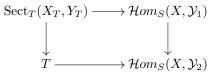\end{array}
$$

commutes. On the other hand, we have that

$$
\left(\mathcal{H}om_S(X, \mathcal{Y}_1) \underset{\mathcal{H}om_S(X, \mathcal{Y}_2)}{\times} T \right)(T') = \left\{ (T' \to T,\ \tau_1' : X_{T'} \to \mathcal{Y}_1,\ \gamma' : F(\tau_1') \simeq \mathrm{pr}_1^* \tau_2) \right\}
$$

by the 2-Yoneda lemma. Thus for a fixed T-scheme T' and a pair (τ_1', γ'), there is a unique $f \in \mathrm{Hom}_{X_T}(X_{T'}, Y_T) = \mathrm{Sect}_T(X_T, Y_T)(T')$ such that

$$
\left(f^*\tau_1,\ f^*\gamma : F(f^*\tau_1) \simeq \mathrm{pr}_1^* \tau_2 \right) \in \left(\mathcal{H}om_S(X, \mathcal{Y}_1) \underset{\mathcal{H}om_S(X, \mathcal{Y}_2)}{\times} T \right)(T')
$$

and there is a unique 2-morphism $(f^*\tau_1, f^*\gamma) \simeq (\tau_1', \gamma')$ induced by ϕ. Therefore we have a Cartesian square

$$
\begin{array}{ccc}
\mathrm{Sect}_T(X_T, Y_T) & \longrightarrow & \mathcal{H}om_S(X, \mathcal{Y}_1) \\
\downarrow & & \downarrow \\
T & \longrightarrow & \mathcal{H}om_S(X, \mathcal{Y}_2)
\end{array}
$$

We have that $Y_T \to X_T$ is finitely presented and quasi-projective (resp. affine). By Lemma 3.1.4 and Theorem 3.1.1, we deduce that $\mathrm{Sect}_T(X_T, Y_T) \to T$ is schematic and locally of finite presentation (resp. affine and of finite presentation). $\qquad\square$

4. Presentation of Bun_G

Recall the definition of Bun_G from Definition 3.0.3. In this section, we prove Theorem 1.0.1. To do this, we embed G in GL_r and reduce to giving a presentation of the moduli stack of locally free modules of rank r.

4.1. **Proving** Bun_G **is algebraic.** The general technique we use to prove results on G-bundles is to reduce to working with GL_r-bundles, which are particularly nice because GL_r-bundles are in fact equivalent to locally free modules, or vector bundles, of rank r.

Lemma 4.1.1. *The morphism from the k-stack*

$$B_r : T \mapsto \{\textit{locally free } \mathcal{O}_T\textit{-modules of rank } r\} + \{\textit{isomorphisms of } \mathcal{O}_T\textit{-modules}\}$$

to $B\mathrm{GL}_r$ sending $\mathcal{E} \mapsto \mathrm{Isom}_T(\mathcal{O}_T^r, \mathcal{E})$ is an isomorphism.

Proof. First observe that B_r is an fpqc stack because QCoh is an fpqc stack [FGI$^+$05, Theorem 4.2.3] and local freeness of rank r persists under fpqc morphisms [EGAIV2, Proposition 2.5.2]. There is a canonical simply transitive right action of GL_r on $\mathrm{Isom}_T(\mathcal{O}_T^r, \mathcal{E})$ by composition. Take a Zariski covering $(T_i \subset T)$ trivializing $\mathcal{E}$. Then $\mathcal{E}$ has a descent datum $(\mathcal{O}_{T_i}^r, g_{ij})$ for $g_{ij} \in \mathrm{GL}_r(T_i \cap T_j)$. Since $\mathrm{Isom}_{T_i}(\mathcal{O}_{T_i}^r, \mathcal{O}_{T_i}^r) \simeq T_i \times \mathrm{GL}_r$, we have a descent datum $(T_i \times \mathrm{GL}_r, g_{ij})$ corresponding to $\mathrm{Isom}_T(\mathcal{O}_T^r, \mathcal{E})$. Conversely, for any $\mathcal{P} \in B\mathrm{GL}_r(T)$, there exists a descent datum $(T_i \times \mathrm{GL}_r, g_{ij})$ for $\mathcal{P}$ over some fppf covering $(T_i \to T)$. For V the standard r-dimensional representation of GL_r, we have that the twist $_{\mathcal{P}}V$ is a module in $B_r(T)$. By construction, $_{\mathcal{P}}V$ has a descent datum $(\mathcal{O}_{T_i}^r, g_{ij})$. By comparing descent data, we deduce that $\mathcal{E} \mapsto \mathrm{Isom}_T(\mathcal{O}_T^r, \mathcal{E})$ is an isomorphism with inverse morphism $\mathcal{P} \mapsto {}_{\mathcal{P}}V$. $\qquad\square$

Note that Lemma 4.1.1 implies that any GL_r-bundle is Zariski locally trivial. We will henceforth implicitly use the isomorphism $B_r \simeq B\mathrm{GL}_r$ to pass between locally free modules and GL_r-bundles.

Fix a base k-scheme S and let $p : X \to S$ be a flat, strongly projective morphism. Fix a very ample invertible sheaf $\mathcal{O}(1)$ on X. Lemma 4.1.1 implies that $\mathrm{Bun}_{\mathrm{GL}_r} \simeq \mathrm{Bun}_r$, where $\mathrm{Bun}_r(T)$ is the groupoid of locally free $\mathcal{O}_{X_T}$-modules of rank r. We say that an $\mathcal{O}_{X_T}$-module $\mathcal{F}$ is *relatively generated by global sections* if the counit of adjunction $p_T^* p_{T*} \mathcal{F} \to \mathcal{F}$ is surjective.

Let $\mathcal{F}$ be a quasi-coherent sheaf on X_T, flat over T. We say that $\mathcal{F}$ is *cohomologically flat over T in degree i* (see [FGI$^+$05, 8.3.10]) if for any Cartesian square

(4.1.1.1)
$$
\begin{array}{ccc}
X_{T'} & \xrightarrow{\ v\ } & X_T \\
{\scriptstyle p_{T'}}\downarrow & & \downarrow{\scriptstyle p_T} \\
T' & \xrightarrow{\ u\ } & T
\end{array}
$$

the canonical morphism $u^* R^i p_{T*} \mathcal{F} \to R^i p_{T'*}(v^* \mathcal{F})$ from [FGI$^+$05, 8.2.19.3] is an isomorphism.

Lemma 4.1.2. *For an S-scheme T, let $\mathcal{F}$ be a quasi-coherent sheaf on X_T, flat over T. If $p_{T*}\mathcal{F}$ is flat and $R^i p_{T*}\mathcal{F} = 0$ for $i > 0$, then $\mathcal{F}$ is cohomologically flat over T in all degrees. For this lemma, we only need p_T to be separated.*

Proof. Consider the Cartesian square (4.1.1.1). To check that $u^* R^i p_{T*} \mathcal{F} \to R^i p_{T'*}(v^* \mathcal{F})$ is an isomorphism, it suffices to restrict to an affine subset of T' lying over an affine subset of T. By the base change formula [FGI$^+$05, Theorem 8.3.2], we have a canonical quasi-isomorphism

$$Lu^* Rp_{T*}\mathcal{F} \overset{\sim}{\to} Rp_{T'*}(v^* \mathcal{F})$$

in the derived category $D(T')$ of $\mathcal{O}_{T'}$-modules. Since $R^i p_{T*}\mathcal{F} = 0$ for $i > 0$, truncation gives a quasi-isomorphism $p_{T*}\mathcal{F} \simeq Rp_{T*}\mathcal{F}$. By flatness of $p_{T*}\mathcal{F}$, we have $Lu^* p_{T*}\mathcal{F} \simeq u^* p_{T*}\mathcal{F}$. Therefore $u^* p_{T*}\mathcal{F} \simeq Rp_{T'*}(v^* \mathcal{F})$ in the derived category, and applying H^i gives

$$u^* R^i p_{T*}\mathcal{F} \simeq R^i p_{T'*}(v^* \mathcal{F}),$$

which is the canonical morphism of [FGI$^+$05, 8.3.2.3]. By [FGI$^+$05, Theorem 8.3.2], this morphism is equal to the base change morphism of [FGI$^+$05, 8.2.19.3]. $\qquad\square$

Proposition 4.1.3. *For an S-scheme T, let*

$$\mathcal{U}_n(T) = \left\{ \begin{array}{l} \mathcal{E} \in \mathrm{Bun}_r(T) \mid R^i p_{T*}(\mathcal{E}(n)) = 0 \ \text{for all } i > 0, \\ \mathcal{E}(n) \ \text{is relatively generated by global sections} \end{array} \right\}$$

be the full subgroupoid of $\mathrm{Bun}_r(T)$. For $\mathcal{E} \in \mathcal{U}_n(T)$, the direct image $p_{T*}(\mathcal{E}(n))$ is flat, and $\mathcal{E}(n)$ is cohomologically flat over T in all degrees. In particular, the inclusion $\mathcal{U}_n \hookrightarrow \mathrm{Bun}_r$ makes $\mathcal{U}_n$ a pseudo-functor.

Proof. For $\mathcal{E} \in \mathcal{U}_n(T)$, let $\mathcal{F} = \mathcal{E}(n)$. To prove $p_{T*}\mathcal{F}$ is flat, we may restrict to an open affine of T. Assuming T is affine, X_T is quasi-compact and separated. Thus we can choose a finite affine cover $\mathfrak{U} = (U_j)_{j=1,\ldots,N}$ of X_T. Since X_T is separated, [EGAIII1, Proposition 1.4.1] implies there is a canonical isomorphism

$$\check{H}^i(\mathfrak{U}, \mathcal{F}) \overset{\sim}{\to} H^i(X_T, \mathcal{F}).$$

Since $R^i p_{T*}\mathcal{F}$ is the quasi-coherent sheaf associated to $H^i(X_T, \mathcal{F})$ by [EGAIII1, Proposition 1.4.10, Corollaire 1.4.11], we deduce that $\check{H}^i(\mathfrak{U}, \mathcal{F}) = 0$ for $i > 0$. Therefore we have an exact sequence

$$0 \to \Gamma(X_T, \mathcal{F}) \to \check{C}^0(\mathfrak{U}, \mathcal{F}) \to \cdots \to \check{C}^{N-1}(\mathfrak{U}, \mathcal{F}) \to 0.$$

Since $\mathcal{F}$ is T-flat, $\check{C}^i(\mathfrak{U}, \mathcal{F}) = \prod_{j_0 < \cdots < j_i} \Gamma(U_{j_0} \cap \cdots \cap U_{j_i}, \mathcal{F})$ is also T-flat. By induction, we conclude that $\Gamma(X_T, \mathcal{F})$ is flat on T. Therefore $p_{T*}\mathcal{F}$ is T-flat. Consequently, Lemma 4.1.2 implies that $\mathcal{F}$ is cohomologically flat over T in all degrees.

Now we wish to show that for a T-scheme T', the pullback $\mathcal{F}_{T'}$ is relatively generated by global sections. This property is local on the base, so we may assume T and T' are affine. In this case, there exists a surjection $\bigoplus \mathcal{O}_{X_T} \twoheadrightarrow \mathcal{F}$, so pulling back gives a surjection $\bigoplus \mathcal{O}_{X_{T'}} \twoheadrightarrow \mathcal{F}_{T'}$, which shows that $\mathcal{F}_{T'}$ is generated by global sections. We conclude that $\mathcal{E}_{T'} \in \mathcal{U}_n(T')$, so $\mathcal{U}_n$ is a pseudo-functor. $\qquad\square$

Remark 4.1.4. Suppose S is affine. Then $\Gamma(S, \mathcal{O}_S)$ is an inductive limit of finitely presented k-algebras, so by [EGAIV3, §8] there exists an affine scheme S_1 of finite presentation over $\mathrm{Spec}\,k$, a morphism $S \to S_1$, and a flat projective morphism $p_1 : X_1 \to S_1$ which is equal to p after base change. We can then define all of our pseudo-functors with respect to $p_1 : X_1 \to S_1$ from $(\mathbf{Sch}_{/S_1})^{\mathrm{op}} \to \mathbf{Gpd}$. The corresponding pseudo-functors for p are then obtained by the

restriction $\mathbf{Sch}_{/S} \to \mathbf{Sch}_{/S_1}$. This will allow us to reduce to the case where the base scheme S is noetherian.

Lemma 4.1.5. *The pseudo-subfunctors* $(\mathcal{U}_n)_{n\in\mathbf{Z}}$ *form an open cover of* Bun_r.

Proof. Take an S-scheme T and a locally free $\mathcal{O}_{X_T}$-module $\mathcal{E} \in \mathrm{Bun}_r(T)$. Then the 2-fibered product $\mathcal{U}_n \times_{\mathrm{Bun}_r} T : (\mathbf{Sch}_{/T})^{\mathrm{op}} \to \mathbf{Gpd}$ sends $T' \to T$ to the equivalence relation[1]

$$\big\{(\mathcal{E}' \in \mathcal{U}_n(T'), \mathcal{E}' \simeq \mathcal{E}_{T'})\big\} + \text{isomorphisms.}$$

Therefore we must prove that for each n, there exists an open subscheme $U_n \subset T$ with the universal property that $T' \to T$ factors through U_n if and only if $R^i p_{T'*}(\mathcal{E}_{T'}(n)) = 0$ in degrees $i > 0$ and $\mathcal{E}_{T'}(n)$ is relatively generated by global sections over T'.

The assertions are Zariski local on S, so by Remark 4.1.4 we can assume that S is the spectrum of a finitely generated k-algebra (and hence an affine noetherian scheme).

We can assume T is affine and express it as a projective limit $\varprojlim T_\lambda$ of spectra of finitely generated $\Gamma(S, \mathcal{O}_S)$-algebras. Since $X \to S$ is proper, X is quasi-compact and separated. The fibered products X_{T_λ} are affine over X. Therefore applying [EGAIV3, Théorème 8.5.2(ii), Proposition 8.5.5] to $X_T = \varprojlim X_{T_\lambda}$, we find that there exists λ and a locally free $\mathcal{O}_{X_{T_\lambda}}$-module $\mathcal{E}_\lambda$ that pulls back to $\mathcal{E}$. Hence $T \to \mathrm{Bun}_r$ factors through T_λ. Replacing T with T_λ, we can assume that T is affine and noetherian.

Set $\mathcal{F} = \mathcal{E}(n)$. Let $U_n \subset T$ be the subset of points $t \in T$ where $H^i(X_t, \mathcal{F}_t) = 0$ for $i > 0$ and $\Gamma(X_t, \mathcal{F}_t) \otimes \mathcal{O}_{X_t} \twoheadrightarrow \mathcal{F}_t$ is surjective. By coherence of $\mathcal{F}$ and Nakayama's lemma, the set of points where $\Gamma(X_t, \mathcal{F}_t) \otimes \mathcal{O}_{X_t} \twoheadrightarrow \mathcal{F}_t$ forms an open subset of T. Since X is quasi-compact, it can be covered by N affines. Base changing, we have that X_t can also be covered by N affines. Since X_t is quasi-compact and separated, $H^i(X_t, \mathcal{F}_t)$ can be computed using Čech cohomology, from which we see that $H^i(X_t, \mathcal{F}_t) = 0$ for $i \geq N$ and all $t \in T$. For fixed i, the set of $t \in T$ where $H^i(X_t, \mathcal{F}_t) = 0$ is an open subset by [Har77, III, Theorem 12.8]. By intersecting finitely many open subsets, we conclude that U_n is an open subset of T. For $t \in U_n$, we have by [Har77, III, Theorem 12.11] that $R^i p_{T*}\mathcal{F} \otimes \kappa(t) = 0$ for $i > 0$. Since p is proper, $R^i p_* \mathcal{F}$ is coherent [EGAIII1, Théorème 3.2.1]. By Nakayama's lemma, this implies that

$$R^i p_{U_n*}(\mathcal{F}_{U_n}) \simeq (R^i p_{T*}\mathcal{F})|_{U_n} = 0.$$

We also have by Nakayama's lemma that $\mathcal{F}_{U_n}$ is relatively generated by global sections over U_n. Therefore $\mathcal{F}_{U_n} \in \mathcal{U}_n(U_n)$.

Now suppose there is a morphism $u : T' \to T$ such that $R^i p_{T'*}\mathcal{F}_{T'} = 0$ for $i > 0$ and $\mathcal{F}_{T'}$ is relatively generated by global sections. Take $t' \in T'$ and set $t = u(t')$. By Proposition 4.1.3, we have that $H^i(X_{t'}, \mathcal{F}_{t'}) = 0$ for $i > 0$. Since $\mathrm{Spec}\,\kappa(t') \to \mathrm{Spec}\,\kappa(t)$ is faithfully flat, [Har77, III, Proposition 9.3] implies that

$$H^i(X_t, \mathcal{F}_t) \underset{\kappa(t)}{\otimes} \kappa(t') \simeq H^i(X_{t'}, \mathcal{F}_{t'}) = 0,$$

and therefore $H^i(X_t, \mathcal{F}_t) = 0$ for $i > 0$. We also have from Proposition 4.1.3 and faithful flatness that $\Gamma(X_{t'}, \mathcal{F}_{t'}) \otimes \mathcal{O}_{X_{t'}} \twoheadrightarrow \mathcal{F}_{t'}$ implies $\Gamma(X_t, \mathcal{F}_t) \otimes \mathcal{O}_{X_t} \twoheadrightarrow \mathcal{F}_t$. This proves that u factors through U_n.

By [Har77, III, Theorem 5.2(b)], there exists an $n \in \mathbf{Z}$ such that $R^i p_{T*}(\mathcal{E}(n)) = 0$ for $i > 0$ and $\mathcal{E}(n)$ is generated by global sections. Therefore the collection $(U_n)_{n\in\mathbf{Z}}$ form an open cover of T. $\qquad\square$

Corollary 4.1.6. *Suppose S is affine and T is an affine scheme mapping to $\mathcal{U}_n$. Then $T \to \mathcal{U}_n$ factors through a scheme locally of finite type over S.*

---

[1]By an equivalence relation, we mean a groupoid whose only automorphisms are the identities.

Proof. From the proof of Lemma 4.1.5, the composition $T \to \mathrm{Bun}_r$ factors through a scheme T_1 of finite type over S. Thus $T \to \mathcal{U}_n$ factors through $\mathcal{U}_n \times_{\mathrm{Bun}_r} T_1$, which is representable by an open subscheme of T_1 by Lemma 4.1.5. $\square$

Remark 4.1.7. For an S-scheme T and $\mathcal{E} \in \mathcal{U}_n(T)$, we claim that the direct image $p_{T*}(\mathcal{E}(n))$ is a locally free $\mathcal{O}_T$-module of finite rank. The claim is Zariski local on T, so we can assume both S and T are affine. We reduce to the case where S is noetherian by Remark 4.1.4. By Corollary 4.1.6, there exists a locally noetherian S-scheme T_1, a Cartesian square

$$
\begin{array}{ccc}
X_T & \xrightarrow{\;v\;} & X_{T_1} \\
{\scriptstyle p_T}\downarrow & & \downarrow{\scriptstyle p_{T_1}} \\
T & \xrightarrow{\;u\;} & T_1
\end{array}
$$

and $\mathcal{E}_1 \in \mathcal{U}_n(T_1)$ such that $v^*\mathcal{E}_1 \simeq \mathcal{E}$. The direct image $p_{T_1*}(\mathcal{E}_1(n))$ is flat and coherent by Proposition 4.1.3 and [EGAIII1, Théorème 3.2.1], hence locally free of finite rank. By cohomological flatness of $\mathcal{E}_1(n)$, we have $p_{T*}(\mathcal{E}(n)) \simeq u^*p_{T_1*}(\mathcal{E}_1(n))$, which proves the claim.

For a polynomial $\Phi \in \mathbf{Q}[\lambda]$, define a pseudo-subfunctor $\mathrm{Bun}_r^{\Phi} \subset \mathrm{Bun}_r$ by letting

$$\mathrm{Bun}_r^{\Phi}(T) = \left\{ \mathcal{E} \in \mathrm{Bun}_r(T) \mid \Phi(m) = \chi(\mathcal{E}_t(m)) \text{ for all } t \in T, m \in \mathbf{Z} \right\}$$

be the full subgroupoid. For a locally noetherian S-scheme T and $\mathcal{E} \in \mathrm{Bun}_r(T)$, the Hilbert polynomial of $\mathcal{E}_t$ is a locally constant function on T by [EGAIII2, Théorème 7.9.4]. Therefore we deduce from Remark 4.1.4 and the proof of Lemma 4.1.5 that the $(\mathrm{Bun}_r^{\Phi})_{\Phi \in \mathbf{Q}[\lambda]}$ form a disjoint open cover of Bun_r. Let

$$\mathcal{U}_n^{\Phi} = \mathcal{U}_n \cap \mathrm{Bun}_r^{\Phi},$$

so that $(\mathcal{U}_n^{\Phi})_{\Phi \in \mathbf{Q}[\lambda]}$ give a disjoint open cover of $\mathcal{U}_n$. For a general S-scheme T and $\mathcal{E} \in \mathcal{U}_n^{\Phi}(T)$, the direct image $p_{T*}(\mathcal{E}(n))$ is locally free of finite rank by Remark 4.1.7. By cohomological flatness of $\mathcal{E}(n)$ and the assumption that $R^i p_{T*}(\mathcal{E}(n)) = 0$ for $i > 0$, we find that $H^0(X_t, \mathcal{E}_t(n)) \simeq p_{T*}(\mathcal{E}(n)) \otimes \kappa(t)$ is a vector space of dimension $\Phi(n)$. Hence $p_{T*}(\mathcal{E}(n))$ is locally free of rank $\Phi(n)$.

Since Bun_r is an fpqc stack, [Sta, Lemma 05UN] shows that all the pseudo-functors defined in the previous paragraph are in fact also fpqc stacks.
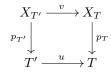
Our goal now is to find schemes with smooth surjective morphisms to the $\mathcal{U}_n^{\Phi}$. To do this, we will introduce a few additional pseudo-functors.

We define the pseudo-functors $\mathcal{Y}_n^{\Phi} : (\mathbf{Sch}_{/S})^{\mathrm{op}} \to \mathbf{Gpd}$. For an S-scheme T, let

$$
\mathcal{Y}_n^{\Phi}(T) = \left\{
\begin{array}{c}
(\mathcal{E}, \phi, \psi) \mid \mathcal{E} \in \mathcal{U}_n^{\Phi}(T),\, \phi : \mathcal{O}_{X_T}^{\Phi(n)} \twoheadrightarrow \mathcal{E}(n) \text{ is surjective,} \\
\text{the adjoint morphism } \psi : \mathcal{O}_T^{\Phi(n)} \to p_{T*}(\mathcal{E}(n)) \text{ is an isomorphism}
\end{array}
\right\},
$$

where a morphism $(\mathcal{E}, \phi, \psi) \to (\mathcal{E}', \phi', \psi')$ is an isomorphism $f : \mathcal{E} \to \mathcal{E}'$ satisfying $f_n \circ \phi = \phi'$, for $f_n = f \otimes \mathrm{id}_{\mathcal{O}(n)} : \mathcal{E}(n) \simeq \mathcal{E}'(n)$. The latter condition is equivalent to $p_{T*}(f_n) \circ \psi = \psi'$, by adjunction. An isomorphism ψ is the same as specifying $\Phi(n)$ elements of $\Gamma(X_T, \mathcal{E}(n))$ that form a basis of $p_{T*}(\mathcal{E}(n))$ as an $\mathcal{O}_T$-module. Note that $\mathcal{Y}_n^{\Phi}(T)$ is an equivalence relation.

Lemma 4.1.8. *Suppose we have a Cartesian square*

$$
\begin{array}{ccc}
X_{T'} & \xrightarrow{\;v\;} & X_T \\
{\scriptstyle p_{T'}}\downarrow & & \downarrow{\scriptstyle p_T} \\
T' & \xrightarrow{\;u\;} & T
\end{array}
$$

Let $\mathcal{M}$ be an $\mathcal{O}_T$-module, $\mathcal{N}$ an $\mathcal{O}_{X_T}$-module, and $\phi : p_T^* \mathcal{M} \to \mathcal{N}$ a morphism of $\mathcal{O}_{X_T}$-modules. If $\psi : \mathcal{M} \to p_{T*} \mathcal{N}$ is the adjoint morphism, then the composition of $u^*(\psi)$ with the base change morphism $u^* p_{T*} \mathcal{N} \to p_{T'*} v^* \mathcal{N}$ corresponds via adjunction of $p_{T'*}, p_{T'}^*$ to $v^*(\phi) : p_{T'}^* u^* \mathcal{M} \simeq v^* p_T^* \mathcal{M} \to v^* \mathcal{N}$.

Proof. Take $v^*(\phi) : v^* p_T^* \mathcal{M} \to v^* \mathcal{N}$. The isomorphism $p_{T'}^* u^* \mathcal{M} \simeq v^* p_T^* \mathcal{M}$ follows by adjunction from the equality $u_* p_{T'*} = p_{T*} v_*$. Therefore it suffices to show that the two morphisms in question are equal after adjunction as morphisms $\mathcal{M} \to p_{T*} v_* v^* \mathcal{N} = u_* p_{T'*} v^* \mathcal{N}$. On the one hand, the adjoint of $v^*(\phi)$ under v_*, v^* is the composition $p_T^* \mathcal{M} \to v_* v^* p_T^* \mathcal{M} \to v_* v^* \mathcal{N}$. By naturality of the unit of adjunction, this morphism is equal to the composition $p_T^* \mathcal{M} \to \mathcal{N} \to v_* v^* \mathcal{N}$. By adjunction under p_{T*}, p_T^*, this morphism corresponds to

$$\mathcal{M} \to p_{T*} \mathcal{N} \to p_{T*} v_* v^* \mathcal{N}.$$

On the other hand, the adjoint of $u^* \mathcal{M} \to u^* p_{T*} \mathcal{N} \to p_{T'*} v^* \mathcal{N}$ under u_*, u^* is equal to $\mathcal{M} \to p_{T*} \mathcal{N} \to u_* p_{T'*} v^* \mathcal{N} = p_{T*} v_* v^* \mathcal{N}$. By definition of the base change morphism [FGI$^+$05, 8.2.19.2-3], the latter morphism $p_{T*} \mathcal{N} \to p_{T*} v_* v^* \mathcal{N}$ is p_{T*} applied to the unit morphism. Hence the two morphisms under consideration are the same. $\qquad\square$

Letting $\mathcal{M} = \mathcal{O}_T^{\Phi(n)}$ and $\mathcal{N} = \mathcal{E}(n)$ in Lemma 4.1.8, we see that the pullbacks of ϕ and ψ are compatible. Thus $\mathcal{Y}_n^\Phi$ has the structure of a pseudo-functor in an unambiguous way.

Lemma 4.1.9. *For $\Psi \in \mathbf{Q}[\lambda]$, define the pseudo-functor $\mathcal{W}^\Psi : (\mathbf{Sch}_{/S})^{\mathrm{op}} \to \mathbf{Gpd}$ by*

$$\mathcal{W}^\Psi(T) = \left\{ (\mathcal{F}, \phi) \mid \mathcal{F} \in \mathrm{Bun}_r^\Psi(T), \phi : \mathcal{O}_{X_T}^{\Psi(0)} \twoheadrightarrow \mathcal{F} \right\}$$

where a morphism $(\mathcal{F}, \phi) \to (\mathcal{F}', \phi')$ is an isomorphism $f : \mathcal{F} \to \mathcal{F}'$ satisfying $f \circ \phi = \phi'$. Then $\mathcal{W}^\Psi$ is representable by a strongly quasi-projective S-scheme.

Proof. Note that the $\mathcal{W}^\Psi(T)$ are equivalence relations, so by considering sets of equivalence classes, $\mathcal{W}^\Psi$ is isomorphic to a subfunctor
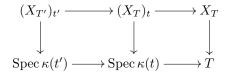
$$W^\Psi \subset \mathrm{Quot}_{\mathcal{O}_X^{\Psi(0)}/X/S}^{\Psi, \mathcal{O}(1)} =: Q.$$

[AK80, Theorem 2.6] shows that Q is representable by a scheme strongly projective over S. We show that $W^\Psi \hookrightarrow Q$ is schematic, open, and finitely presented, which will imply that $\mathcal{W}^\Psi \simeq W^\Psi$ is representable by a strongly quasi-projective S-scheme. The claim is Zariski local on S, so by [EGAG, 0, Corollaire 4.5.5] and Remark 4.1.4, we can assume S is noetherian.

Let T be an S-scheme and $(\mathcal{F}, \phi) \in Q(T)$. Let $U \subset X_T$ be the set of points x where the stalk $\mathcal{F} \otimes \mathcal{O}_{X_T, x}$ is free. Since $\mathcal{F}$ is coherent, U is an open subset. In particular, it is the maximal open subset such that $\mathcal{F}|_U$ is locally free. We claim that the open subset

$$T - p_T(X_T - U) \subset T$$

represents the fibered product $W^\Psi \times_Q T$. This is equivalent to saying that a morphism $T' \to T$ lands in $T - p_T(X_T - U)$ if and only if $\mathcal{F}_{T'}$ is locally free. If $T' \to T$ lands in $T - p_T(X_T - U)$, then $X_{T'} \to X_T$ lands in U, which implies $\mathcal{F}_{T'}$ is locally free. Conversely, suppose we start with a morphism $T' \to T$ such that $\mathcal{F}_{T'}$ is locally free. Assume for the sake of contradiction that there exists $t' \in T$ and $x \in X_T$ that morphism to the same point $t \in T$ such that $\mathcal{F} \otimes \mathcal{O}_{X_T, x}$ is not free. We have Cartesian squares

$$
\begin{array}{ccc}
(X_{T'})_{t'} & \longrightarrow (X_T)_t & \longrightarrow X_T \\
\downarrow & \downarrow & \downarrow \\
\mathrm{Spec}\,\kappa(t') & \longrightarrow \mathrm{Spec}\,\kappa(t) & \longrightarrow T
\end{array}
$$

where $\operatorname{Spec}\kappa(t') \to \operatorname{Spec}\kappa(t)$ is faithfully flat. Then $\mathcal{F}_{t'} \simeq \mathcal{F}_t \otimes_{\kappa(t)} \kappa(t')$ is a flat $\mathcal{O}_{X_{t'}}$-module. By faithful flatness, this implies that $\mathcal{F}_t$ is flat over $\mathcal{O}_{X_t}$. Recall from the definition of Q that $\mathcal{F}$ is assumed to be T-flat. Therefore $\mathcal{F} \otimes \mathcal{O}_{X_T,x}$ is $\mathcal{O}_{T,t}$-flat and $\mathcal{F}_t \otimes \mathcal{O}_{X_t,x}$ is $\mathcal{O}_{X_t,x}$-flat. By [Mat80, 20.G], we conclude that $\mathcal{F} \otimes \mathcal{O}_{X_T,x}$ is $\mathcal{O}_{X_T,x}$-flat and hence a free module. We therefore have a contradiction, so $T' \to T$ must factor through $T - p_T(X_T - U)$. Letting $T = Q$, we have shown that W^Ψ is representable by an open subscheme of Q. Since S is noetherian, Q is also noetherian. Therefore $W^\Psi \hookrightarrow Q$ is finitely presented. $\qquad\square$

Lemma 4.1.10. *The pseudo-functor $\mathcal{Y}_n^\Phi$ is representable by a scheme Y_n^Φ which is strongly quasi-projective over S.*

Proof. Define the polynomial $\Psi \in \mathbf{Q}[\lambda]$ by $\Psi(\lambda) = \Phi(\lambda + n)$. Let $\mathcal{W}^\Psi$ be the pseudo-functor of Lemma 4.1.9. Then there is a morphism $\mathcal{W}^\Psi \to \operatorname{Bun}_r^\Phi$ by sending $(\mathcal{F}, \phi) \mapsto \mathcal{F}(-n)$. The corresponding 2-fibered product $\mathcal{U}_n^\Phi \times_{\operatorname{Bun}_r^\Phi} \mathcal{W}^\Psi$ is isomorphic to the pseudo-functor $\mathcal{Z}_n^\Phi :$ $(\mathbf{Sch}_{/S})^{\mathrm{op}} \to \mathbf{Gpd}$ defined by

$$\mathcal{Z}_n^\Phi(T) = \left\{ (\mathcal{E}, \phi) \mid \mathcal{E} \in \mathcal{U}_n^\Phi(T), \phi : \mathcal{O}_{X_T}^{\Phi(n)} \twoheadrightarrow \mathcal{E}(n) \right\}$$

where a morphism $(\mathcal{E}, \phi) \to (\mathcal{E}', \phi')$ is an isomorphism $f : \mathcal{E} \to \mathcal{E}'$ satisfying $f_n \circ \phi = \phi'$. By Lemma 4.1.5, the morphism $\mathcal{Z}_n^\Phi \to \mathcal{W}^\Psi$ is schematic and open, so Lemma 4.1.9 implies that $\mathcal{Z}_n^\Phi$ is representable by an open subscheme Z_n^Φ of W^Ψ. We claim that $Z_n^\Phi \hookrightarrow W^\Psi$ is finitely presented. By Remark 4.1.4, we can reduce to the case where S is noetherian, and now the assertion is trivial. Therefore Z_n^Φ is strongly quasi-projective over S.

Next we show that the morphism $\mathcal{Y}_n^\Phi \to \mathcal{Z}_n^\Phi$ sending $(\mathcal{E}, \phi, \psi) \mapsto (\mathcal{E}, \phi)$ is schematic, open, and finitely presented. By [EGAG, 0, Corollaire 4.5.5] and Remark 4.1.4, we assume S is noetherian. Take $(\mathcal{E}, \phi) \in \mathcal{Z}_n^\Phi(T)$ and let ψ be the adjoint morphism $\mathcal{O}_T^{\Phi(n)} \to p_{T*}(\mathcal{E}(n))$. Define the open subscheme $U \subset T$ to be the complement of the support of $\operatorname{coker}(\psi)$. By Nakayama's lemma and faithful flatness of field extensions, a morphism $u : T' \to T$ lands in U if and only if the pullback $u^*(\psi)$ is a surjection. Since $\mathcal{O}_T^{\Phi(n)}$ and $p_{T*}(\mathcal{E}(n))$ are locally free $\mathcal{O}_T$-modules of the same rank, $u^*(\psi)$ is surjective if and only if it is an isomorphism. Recall from Proposition 4.1.3 that the base change morphism $u^* p_{T*}(\mathcal{E}(n)) \to p_{T'*}(\mathcal{E}_{T'}(n))$ is an isomorphism. Thus the compatibility assertion of Lemma 4.1.8 shows that $u^*(\psi)$ is an isomorphism if and only if the adjoint of $v^*(\phi)$ is an isomorphism, for $v : X_{T'} \to X_T$. Thus U represents the fibered product $\mathcal{Y}_n^\Phi \times_{\mathcal{Z}_n^\Phi} T$. Taking $T = Z_n^\Phi$, which is noetherian, we deduce that $\mathcal{Y}_n^\Phi \to \mathcal{Z}_n^\Phi$ is schematic, open, and finitely presented. Therefore $\mathcal{Y}_n^\Phi$ is representable by a strongly quasi-projective S-scheme. $\qquad\square$

Lemma 4.1.11. *There is a canonical right $\operatorname{GL}_{\Phi(n)}$-action on Y_n^Φ such that the morphism $Y_n^\Phi \to \mathcal{U}_n^\Phi$ sending $(\mathcal{E}, \phi, \psi) \mapsto \mathcal{E}$ is a $\operatorname{GL}_{\Phi(n)}$-bundle. Lemma 2.1.1 gives an isomorphism*
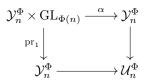
$$\mathcal{U}_n^\Phi \simeq [Y_n^\Phi / \operatorname{GL}_{\Phi(n)}]$$

sending $\mathcal{E} \in \mathcal{U}_n^\Phi(T)$ to a $\operatorname{GL}_{\Phi(n)}$-equivariant morphism $\operatorname{Isom}_T\left(\mathcal{O}_T^{\Phi(n)}, p_{T*}(\mathcal{E}(n))\right) \to Y_n^\Phi$.
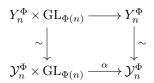
Proof. We define an action morphism $\alpha : \mathcal{Y}_n^\Phi \times \operatorname{GL}_{\Phi(n)} \to \mathcal{Y}_n^\Phi$ by sending

$$((\mathcal{E}, \phi, \psi), g) \mapsto (\mathcal{E}, \phi \circ p_T^*(g), \psi \circ g)$$

over an S-scheme T. Then the diagram

$$
\begin{array}{ccc}
\mathcal{Y}_n^\Phi \times \operatorname{GL}_{\Phi(n)} & \xrightarrow{\ \alpha\ } & \mathcal{Y}_n^\Phi \\
{\scriptstyle \operatorname{pr}_1}\big\downarrow & & \big\downarrow \\
\mathcal{Y}_n^\Phi & \longrightarrow & \mathcal{U}_n^\Phi
\end{array}
$$

is 2-commutative where the 2-morphism is just the identity. Then α induces a right $\mathrm{GL}_{\Phi(n)}$-action on Y_n^Φ by taking equivalence classes, i.e., we have a 2-commutative square

$$\begin{array}{ccc} Y_n^\Phi \times \mathrm{GL}_{\Phi(n)} & \longrightarrow & Y_n^\Phi \\ \sim\downarrow & & \downarrow\sim \\ \mathcal{Y}_n^\Phi \times \mathrm{GL}_{\Phi(n)} & \stackrel{\alpha}{\longrightarrow} & \mathcal{Y}_n^\Phi \end{array}$$

and the associated 2-morphism must satisfy the associativity condition of 2.1.3(2) because $\mathcal{Y}_n^\Phi(T)$ is an equivalence relation. We deduce that $Y_n^\Phi \to \mathcal{U}_n^\Phi$ is $\mathrm{GL}_{\Phi(n)}$-invariant.

For an S-scheme T, let $\mathcal{E} \in \mathcal{U}_n^\Phi(T)$. Then the fibered product on $(\mathbf{Sch}_{/T})^{\mathrm{op}} \to \mathbf{Gpd}$ is given by

$$\left(\mathcal{Y}_n^\Phi \underset{\mathcal{U}_n^\Phi}{\times} T\right)(T') = \left\{(\mathcal{N}, \phi, \psi, \gamma) \mid (\mathcal{N}, \phi, \psi) \in \mathcal{Y}_n^\Phi, \gamma : \mathcal{N} \simeq \mathcal{E}_{T'}\right\}$$

where a morphism $(\mathcal{N}, \phi, \psi, \gamma) \to (\mathcal{N}', \phi', \psi', \gamma')$ is an isomorphism $f : \mathcal{N} \to \mathcal{N}'$ such that $f_n \circ \phi = \phi'$ and $\gamma' \circ f = \gamma$. Observe that $(\mathcal{Y}_n^\Phi \times_{\mathcal{U}_n^\Phi} T)(T')$ is an equivalence relation, and $\mathcal{Y}_n^\Phi \times_{\mathcal{U}_n^\Phi} T$ is isomorphic to the functor $\mathcal{P} : (\mathbf{Sch}_{/T})^{\mathrm{op}} \to \mathbf{Set}$ sending

$$(T' \to T) \mapsto \left\{(\phi, \psi) \mid \phi : \mathcal{O}_{X_{T'}}^{\Phi(n)} \twoheadrightarrow \mathcal{E}_{T'}(n), \text{ the adjoint } \psi \text{ is an isomorphism}\right\}.$$

In fact, the surjectivity of ϕ is automatic: since $\mathcal{E}_{T'}(n)$ is relatively generated by global sections, if we have an isomorphism $\psi : \mathcal{O}_{T'}^{\Phi(n)} \simeq p_{T'*}(\mathcal{E}_{T'}(n))$, then the adjoint morphism

$$\phi : \mathcal{O}_{X_{T'}}^{\Phi(n)} \simeq p_{T'}^* p_{T'*}(\mathcal{E}_{T'}(n)) \twoheadrightarrow \mathcal{E}_{T'}(n)$$

is surjective. Therefore $\mathcal{P}(T')$ is just equal to a choice of basis for $p_{T'*}(\mathcal{E}_{T'}(n))$. The induced right action of $\mathrm{GL}_{\Phi(n)}$ on $\mathcal{P}$ is defined by $\psi.g = \psi \circ g$ for $g \in \mathrm{GL}_{\Phi(n)}(T')$. By Remark 4.1.7, the direct image $p_{T*}(\mathcal{E}(n))$ is locally free of rank $\Phi(n)$. By cohomological flatness of $\mathcal{E}(n)$ over T (see Proposition 4.1.3), we have an isomorphism $\mathcal{P} \simeq \mathrm{Isom}_T\left(\mathcal{O}_T^{\Phi(n)}, p_{T*}(\mathcal{E}(n))\right)$. The latter is the right $\mathrm{GL}_{\Phi(n)}$-bundle associated to $p_{T*}(\mathcal{E}(n))$ by Lemma 4.1.1. $\qquad\square$

In fact, we can say a little more about the structure of Y_n^Φ, and this will be useful in §5.

Lemma 4.1.12. *There exists a $\mathrm{GL}_{\Phi(n)}$-equivariant invertible sheaf on Y_n^Φ that is very ample over S.*

Proof. Define the polynomial $\Psi(\lambda) = \Phi(\lambda+n)$ and let $Q = \mathrm{Quot}_{\mathcal{O}_X^{\Phi(n)}/X/S}^{\Psi, \mathcal{O}(1)}$. By [AK80, Theorem 2.6], there exists an integer m and a closed immersion

$$Q \hookrightarrow \mathbf{P}\left(\bigwedge^{\Psi(m)} p_*(\mathcal{O}_X^{\Phi(n)}(m))\right).$$

Let $\mathcal{O}_Q(1)$ be the very ample invertible $\mathcal{O}_Q$-module corresponding to this immersion. [AK80, Theorem 2.6] also implies that for a morphism $q : T \to Q$ corresponding to $\mathcal{O}_{X_T}^{\Phi(n)} \twoheadrightarrow \mathcal{F}$, we have a canonical isomorphism $q^*(\mathcal{O}_Q(1)) \simeq \bigwedge^{\Psi(m)} p_{T*}(\mathcal{F}(m))$. From Lemmas 4.1.9 and 4.1.10, we have an immersion

$$Y_n^\Phi \hookrightarrow Q : (\mathcal{E}, \phi, \psi) \mapsto (\phi : \mathcal{O}_{X_T}^{\Phi(n)} \twoheadrightarrow \mathcal{E}(n)).$$

Thus for any morphism $y : T \to Y_n^\Phi$ corresponding to $(\mathcal{E}, \phi, \psi)$, there exists a canonical isomorphism $y^*(\mathcal{O}_Q(1)|_{Y_n^\Phi}) \simeq \bigwedge^{\Phi(n+m)} p_{T*}(\mathcal{E}(n+m))$. Therefore $y^*(\mathcal{O}_Q(1)|_{Y_n^\Phi})$ only depends on the image of y in $\mathcal{U}_n^\Phi(T)$. Since $Y_n^\Phi \to \mathcal{U}_n^\Phi$ is $\mathrm{GL}_{\Phi(n)}$-invariant, it follows that for any $g \in \mathrm{GL}_{\Phi(n)}(T)$,

there is a canonical isomorphism $y^*(\mathcal{O}_Q(1)|_{Y_n^\Phi}) \simeq (y.g)^*(\mathcal{O}_Q(1)|_{Y_n^\Phi})$. This gives the very ample invertible sheaf $\mathcal{O}_Q(1)|_{Y_n^\Phi}$ a $\mathrm{GL}_{\Phi(n)}$-equivariant structure. □

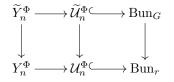*Proof of Theorem 1.0.1.* We know the diagonal of Bun_G is schematic, affine, and finitely presented by Corollary 3.2.2. For any affine algebraic group G, there exists an integer r such that $G \subset \mathrm{GL}_r$ is a closed subgroup [DG70, II, §2, Théorème 3.3].

First, we prove the theorem for GL_r. Lemma 4.1.11 implies that the morphisms $\mathcal{Y}_n^\Phi \to \mathcal{U}_n^\Phi$ are smooth and surjective. By taking an open covering of S and using [EGAG, 0, Corollaire 4.5.5] and Lemma 4.1.10, we see that each $\mathcal{Y}_n^\Phi$ is representable by a scheme Y_n^Φ of finite presentation over S. The morphism

$$Y = \bigsqcup_{n \in \mathbf{Z}, \Phi \in \mathbf{Q}[\lambda]} Y_n^\Phi \to \mathrm{Bun}_r$$

is then smooth and surjective by Lemma 4.1.5, where Y is locally of finite presentation over S. Therefore Bun_r is an algebraic stack locally of finite presentation over S. It follows from [Sta, Lemma 05UN] that all $\mathcal{U}_n^\Phi \subset \mathrm{Bun}_r$ are algebraic stacks, and they are of finite presentation over S since the Y_n^Φ are.

By Corollary 3.2.4, the morphism $\mathrm{Bun}_G \to \mathrm{Bun}_r$ corresponding to $G \subset \mathrm{GL}_r$ is schematic and locally of finite presentation. Consider the Cartesian squares

$$
\begin{array}{ccccc}
\widetilde{Y}_n^\Phi & \longrightarrow & \widetilde{\mathcal{U}}_n^\Phi & \hookrightarrow & \mathrm{Bun}_G \\
\downarrow & & \downarrow & & \downarrow \\
Y_n^\Phi & \longrightarrow & \mathcal{U}_n^\Phi & \hookrightarrow & \mathrm{Bun}_r
\end{array}
$$

where by base change, the morphisms $\widetilde{Y}_n^\Phi \to \widetilde{\mathcal{U}}_n^\Phi$ are smooth and surjective, and the $\widetilde{\mathcal{U}}_n^\Phi$ form an open covering of Bun_G. From Theorem 3.1.1 and Corollary 3.2.4, we deduce that $\widetilde{Y}_n^\Phi$ is representable by a disjoint union of schemes of finite presentation over S. Applying [Sta, Lemma 05UP] to the smooth surjective morphism $\widetilde{Y}_n^\Phi \to \widetilde{\mathcal{U}}_n^\Phi$, we conclude that Bun_G is covered by open substacks of finite presentation over S. □

4.2. **Examples.** We can say a little more about the properties of Bun_G after imposing extra conditions on G and $X \to S$. In this subsection we discuss a few such examples.

4.2.1. *Case of a curve.* In the following example, assume $S = \mathrm{Spec}\, k$ and k is algebraically closed. Let X be a smooth projective integral scheme of dimension 1 over k. We mention some examples concerning the previous constructions in this situation.

First we show that for a locally free $\mathcal{O}_X$-module $\mathcal{E}$ of rank r, the Hilbert polynomial is determined by $\deg \mathcal{E}$. Let $K(X)$ denote the Grothendieck group of X. By [Man69, Corollary 10.8], the homomorphism

$$K(X) \to \mathrm{Pic}(X) \oplus \mathbf{Z}$$

sending a coherent sheaf $\mathcal{F} \in \mathrm{Coh}(X)$ to $(\det \mathcal{F}, \mathrm{rk}\, \mathcal{F})$ is an isomorphism, and the inverse homomorphism sends $(\mathcal{L}, m) \mapsto [\mathcal{L}] + (m-1)[\mathcal{O}_X]$. Therefore $[\mathcal{E}] = [\det \mathcal{E}] + (r-1)[\mathcal{O}_X]$ in $K(X)$. Tensoring by $\mathcal{O}(n)$, we deduce that

$$[\mathcal{E}(n)] = [(\det \mathcal{E}) \otimes \mathcal{O}(n)] + (r-1)[\mathcal{O}(n)].$$

Applying $\deg$ to this equality, we find that $\deg \mathcal{E}(n) = \deg \mathcal{E} + r \deg \mathcal{O}(n)$, or equivalently $\chi(\mathcal{E}(n)) = \deg \mathcal{E} + r\chi(\mathcal{O}(n))$. Therefore Bun_r is a disjoint union of $(\mathrm{Bun}_r^d)_{d \in \mathbf{Z}}$, the substacks of locally free sheaves of rank r which are fiber-wise of degree d, and the previous results hold with Bun_r^Φ replaced by Bun_r^d. We also note that since $\dim X = 1$, for any $\mathcal{O}_X$-module $\mathcal{F}$, the higher

cohomologies $H^i(X, \mathcal{F}) = 0$ for $i > 1$ by [Tôhoku, Théorème 3.6.5]. Therefore the substacks $\mathcal{U}_n$ are characterized by relative generation by global sections and vanishing of $R^1 p_{T*}$.

Theorem 1.0.1 says that Bun_r is locally of finite type over k, so the stack is a disjoint union of connected components (take the images of the connected components of a presentation using [Sta, Lemma 05UP]). By considering degrees, Bun_r has infinitely many connected components, so it is not quasi-compact. One might ask if the connected components of Bun_r are quasi-compact. The following example shows that this should not be expected to be the case in general.

Example 4.2.2. There exists a connected component of Bun_2 that is not quasi-compact. First we show that there exists n_0 such that $\mathcal{O}_X^2$ and $\mathcal{O}(-n) \oplus \mathcal{O}(n) \in \text{Bun}_2(k)$ all lie in the same connected component of Bun_2 for $n \geq n_0$. By Serre's Theorem [Har77, II, Theorem 5.17], there exists n_0 such that $\mathcal{O}_X^2(n)$ is generated by finitely many global sections for $n \geq n_0$. Fix such an n. Using the fact that $\dim X = 1$, it follows from a lemma of Serre [Mum66, pg. 148] that there exists a section $s \in \Gamma(X, \mathcal{O}_X^2(n))$ that is nonzero in $\mathcal{O}_X^2(n) \otimes \kappa(x)$ for all closed points $x \in X(k)$. This gives a short exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{O}(-n) \to \mathcal{O}_X^2 \to \mathcal{L} \to 0$$

where $\mathcal{L}$ is locally free of rank 1. From the definition of the determinant, we have

$$\mathcal{L} \simeq \det \mathcal{L} \simeq \bigwedge^2 (\mathcal{O}_X^2) \otimes \mathcal{O}(n) \simeq \mathcal{O}(n).$$

Therefore $\mathcal{O}_X^2$ is an extension of $\mathcal{O}(n)$ by $\mathcal{O}(-n)$. By [Tôhoku, Corollaire 4.2.3], we have $\text{Ext}^1_{\mathcal{O}_X}(\mathcal{O}(n), \mathcal{O}(-n)) \simeq H^1(X, \mathcal{O}(-2n))$, which is a finite dimensional k-vector space [Har77, III, Theorem 5.2]. Let

$$\mathbf{V} = \text{Spec} \, \text{Sym}_k(H^1(X, \mathcal{O}(-2n))^\vee).$$

For an affine noetherian k-scheme $T = \text{Spec} \, A$, we have $\text{Hom}_k(T, \mathbf{V}) \simeq H^1(X, \mathcal{O}(-2n)) \otimes_k A$. By cohomology and flat base change [Har77, III, Proposition 9.3], this gives

$$\text{Hom}_k(T, \mathbf{V}) \simeq H^1(X_T, \mathcal{O}_{X_T}(-2n)) \simeq \text{Ext}^1_{\mathcal{O}_{X_T}}(\mathcal{O}_{X_T}(n), \mathcal{O}_{X_T}(-n)).$$

The identity morphism $\text{id}_{\mathbf{V}}$ corresponds to some extension of $\mathcal{O}_{X_{\mathbf{V}}}(n)$ by $\mathcal{O}_{X_{\mathbf{V}}}(-n)$, which must be a locally free $\mathcal{O}_{X_{\mathbf{V}}}$-module of rank 2. This defines a morphism $\mathbf{V} \to \text{Bun}_2$. We have that $\mathcal{O}_X^2$ and $\mathcal{O}(-n) \oplus \mathcal{O}(n)$ both correspond to k-points of $\mathbf{V}$. Since $\mathbf{V}$ is connected, this implies that these two points lie in the same connected component of Bun_2. Now suppose there exists a quasi-compact scheme Y° with a surjective morphism to the connected component of Bun_2 in question. The morphism $Y^\circ \to \text{Bun}_2$ corresponds to a locally free sheaf $\mathcal{E}$ of rank 2 on X_{Y°. We can assume Y° is noetherian by [EGAIV3, §8]. Then by [Har77, II, Theorem 5.17], there exists an integer $n \geq n_0$ such that $\mathcal{E}(n)$ is relatively generated by global sections. By Nullstellensatz, there must exists a k-point of Y° mapping to the isomorphism class of $\mathcal{O}(-n-1) \oplus \mathcal{O}(n+1) \in \text{Bun}_2(k)$. Therefore we have that $\mathcal{O}(-1) \oplus \mathcal{O}(2n+1)$ is generated by global sections. In particular, $\mathcal{O}(-1)$ is generated by global sections, which is a contradiction since $\Gamma(X, \mathcal{O}(-1)) = 0$ by [Har77, IV, Lemma 1.2, Corollary 3.3].

4.2.3. *Picard scheme and stack.* In the previous example we considered locally free sheaves of rank 2 on a dimension 1 scheme. As the next example shows, if we consider locally free sheaves of rank 1, then the connected components of Bun_1 will in fact be quasi-compact. Assume for the moment that $X \to S = \text{Spec} \, k$ is a curve as in §4.2.1.

Example 4.2.4. Let g be the genus of X. Then for any integer d, we have $\mathcal{U}_n^d = \text{Bun}_1^d$ for $n \geq 2g - d$. Take $Y \to \text{Bun}_1$ a smooth surjective morphism with Y a scheme locally of finite

type over k. It suffices to show that $\mathcal{U}_n^d \times_{\mathrm{Bun}_1^d} Y \hookrightarrow Y$ is surjective. Since k is algebraically closed, it is enough to check surjectivity on k-points. Suppose we have $\mathcal{L} \in \mathrm{Bun}_1^d(k)$. Then $\deg \mathcal{L}(n) \geq 2g$ since $\deg \mathcal{O}(1) > 0$ by [Har77, IV, Corollary 3.3]. Therefore [Har77, IV, Example 1.3.4, Corollary 3.2] and Serre duality imply that $H^1(X, \mathcal{L}(n)) = 0$ and $\mathcal{L}(n)$ is generated by global sections. We conclude that $\mathcal{U}_n^d = \mathrm{Bun}_1^d$ for $n \geq 2g - d$, and in particular, Bun_1^d is quasi-compact.

Now suppose $p : X \to S$ is a separated morphism of finite type between schemes such that the fppf sheaf $\mathrm{Pic}_{X/S}$ is representable by a scheme. This is satisfied, for example, when S is locally noetherian and p is flat and locally projective with geometrically integral fibers [FGI$^+$05, Theorem 9.4.8]. Additionally assume that the unit morphism $\mathcal{O}_T \to p_{T*}\mathcal{O}_{X_T}$ is an isomorphism for all S-schemes T. By [FGI$^+$05, Exercise 9.3.11], this holds when S is locally noetherian and p is proper and flat with geometric fibers that are reduced and connected.
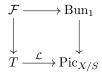
The stack Bun_1 has the special property that the associated coarse space (see [LMB00, Remarque 3.19]) is the Picard scheme $\mathrm{Pic}_{X/S}$. We will use properties of $\mathrm{Pic}_{X/S}$ to deduce properties of Bun_1. We thank Thanos D. Papaïoannou for suggesting this approach.

Proposition 4.2.5. *Suppose $X \to S$ admits a section. Then there is an isomorphism*

$$B\mathbf{G}_m \times \mathrm{Pic}_{X/S} \to \mathrm{Bun}_1$$

over S such that the morphism to the coarse space is the second projection.

Proof. Since p has a section, we have that $\mathrm{Pic}_{X/S}(T) = \mathrm{Pic}(X_T)/\mathrm{Pic}(T)$ by [FGI$^+$05, Theorem 9.2.5]. For an invertible sheaf $\mathcal{L} \in \mathrm{Pic}(X_T)$, we consider the Cartesian square

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathrm{Bun}_1 \\ \downarrow & & \downarrow \\ T & \xrightarrow{\ \mathcal{L}\ } & \mathrm{Pic}_{X/S} \end{array}$$

where $\mathcal{F} : (\mathbf{Sch}_{/T})^{\mathrm{op}} \to \mathbf{Gpd}$ is the stack sending

$$(T' \to T) \mapsto \{\mathcal{M} \in \mathrm{Pic}(X_{T'}) \mid \text{there exists } \mathcal{N} \in \mathrm{Pic}(T'),\ \mathcal{M} \simeq \mathcal{L}_{T'} \otimes p_{T'}^*\mathcal{N}\}$$

and a morphism $\mathcal{M} \to \mathcal{M}'$ is an isomorphism of $\mathcal{O}_{X_{T'}}$-modules. We define the morphism $B\mathbf{G}_m \times T \to \mathcal{F}$ by sending an invertible $\mathcal{O}_{X_{T'}}$-module $\mathcal{N}$ to $\mathcal{L}_{T'} \otimes p_{T'}^*\mathcal{N}$. Now [FGI$^+$05, Lemma 9.2.7] shows that $p_{T'}^*$ is fully faithful from the category of locally free sheaves of finite rank on T' to that on $X_{T'}$. Therefore the previous morphism is an isomorphism. Taking $T = \mathrm{Pic}_{X/S}$ and considering the identity morphism on $\mathrm{Pic}_{X/S}$ proves the claim. $\qquad\square$

Corollary 4.2.6. *The morphism $\mathrm{Bun}_1 \to \mathrm{Pic}_{X/S}$ is smooth, surjective, and of finite presentation.*

Proof. The claim is fppf local on S, so we assume S is connected and X nonempty. The image of p is open and closed since p is flat and proper, and hence p is an fppf morphism. The base change $X \times_S X \to X$ admits a section via the diagonal. Then Proposition 4.2.5 implies that the morphism $\mathrm{Bun}_1 \times_S X \to \mathrm{Pic}_{X \times_S X/X}$ is smooth, surjective, and of finite presentation because $B\mathbf{G}_m \to \cdot$ has these properties ($\cdot \to B\mathbf{G}_m$ is a smooth presentation by Lemma 2.5.2 since $\mathbf{G}_m$ is smooth). $\qquad\square$

For $\mathcal{L} \in \mathrm{Bun}_1(T)$, the automorphisms of $\mathcal{L}$ are given by $\Gamma(X_T, \mathcal{O}_{X_T})^\times \simeq \Gamma(T, \mathcal{O}_T^\times)$ by our assumption on $\mathcal{O}_T \to p_{T*}\mathcal{O}_{X_T}$. Thus $\mathrm{Isom}(\mathcal{L}, \mathcal{L}) \simeq \mathbf{G}_m \times T$. We note that the smoothness and surjectivity of $\mathrm{Bun}_1 \to \mathrm{Pic}_{X/S}$ then follow from the proof of [LMB00, Corollaire 10.8].

Example 4.2.7. Let S be locally noetherian and $X \to S$ flat and projective with geometrically integral fibers. Then the connected components of Bun_1 are of finite type over S. Indeed, the connected components of $\mathrm{Pic}_{X/S}$ are of finite type over S by [FGI$^+$05, Theorem 9.6.20], so their preimages under $\mathrm{Bun}_1 \to \mathrm{Pic}_{X/S}$ are also of finite type over S by Corollary 4.2.6.

5. Level structure

Let $p : X \to S$ be a separated morphism of schemes over k. Suppose we have a section $x : S \to X$ over S. Then the graph $\Gamma_x : S \to S \times_S X$ is a closed immersion, which implies x is a closed immersion. Let $\mathcal{I}_x$ be the corresponding ideal sheaf. For any positive integer n, define $i_n : (nx) \hookrightarrow X$ to be the closed immersion corresponding to the ideal sheaf $\mathcal{I}^n$. We assume that X is infinitesimally flat in x (see [Jan03, I, 7.4]), i.e., we require that $\pi_n : (nx) \to S$ is finite and locally free for all n.

Remark 5.0.8. We discuss two cases where the condition of infinitesimal flatness is automatic. First, suppose S is the spectrum of a field. Then x is a closed point, and $(nx) = \mathrm{Spec}\, \mathcal{O}_{X,x}/\mathfrak{m}_x^n$ is the spectrum of an artinian ring, where $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{O}_{X,x}$. Thus (nx) is finite and free over S.

Next suppose that x lands in the smooth locus $X_{\mathrm{sm}} \subset X$ of $p : X \to S$. Pick a point $s \in S$. By assumption, p is smooth at $x(s)$. Therefore by [EGAIV4, Théorème 17.12.1], there exists an open neighborhood U of s where $\mathcal{I}_x/\mathcal{I}_x^2|_U$ is a locally free $\mathcal{O}_U$-module, and the canonical morphism $\mathrm{Sym}^\bullet_{\mathcal{O}_U}(\mathcal{I}_x/\mathcal{I}_x^2|_U) \to \mathcal{G}r^\bullet(x|_U)$ is an isomorphism. This is true for all $s \in S$, so we can take $U = S$ in the above. Now we have short exact sequences of $\mathcal{O}_X$-modules

$$0 \to x_*(\mathcal{G}r^n(x)) \to \mathcal{O}_X/\mathcal{I}_x^{n+1} \to \mathcal{O}_X/\mathcal{I}_x^n \to 0,$$

where $\mathcal{G}r^n(x)$ is a locally free $\mathcal{O}_S$-module. We know that $\mathcal{O}_X/\mathcal{I}_x \simeq x_*\mathcal{O}_S$ is a free $\mathcal{O}_S$-module. By induction we find that the above sequence is locally split exact, and hence $\mathcal{O}_X/\mathcal{I}_x^n$ is a locally free $\mathcal{O}_S$-module for all n. The closed immersion $S \simeq (x) \hookrightarrow (nx)$ is an isomorphism on topological spaces, so we deduce that $(nx) \to S$ is finite and locally free.

Define the pseudo-functor $\mathrm{Bun}_G^{(nx)} : (\mathbf{Sch}_{/S})^{\mathrm{op}} \to \mathbf{Gpd}$ by

$$\mathrm{Bun}_G^{(nx)}(T) = \left\{ (\mathcal{P}, \phi) \mid \mathcal{P} \in \mathrm{Bun}_G(T),\, \phi \in \mathrm{Hom}_{BG((nx)_T)}((nx)_T \times G, i_n^*\mathcal{P}) \right\}$$

where a morphism $(\mathcal{P}, \phi) \to (\mathcal{P}', \phi')$ is a G-equivariant morphism $f : \mathcal{P} \to \mathcal{P}'$ over X_T such that $i_n^*(f) \circ \phi = \phi'$. Since BG and Bun_G are fpqc stacks, we observe that $\mathrm{Bun}_G^{(nx)}$ is also an fpqc stack.

The main result of this section is the following:

Theorem 5.0.9. *Suppose $p : X \to S$ is a flat, finitely presented, projective morphism with geometrically integral fibers. Let $\mathcal{V} \subset \mathrm{Bun}_G$ be a quasi-compact open substack. There exists an integer N_0 such that for all $N \geq N_0$, the 2-fibered product $\mathrm{Bun}_G^{(Nx)} \times_{\mathrm{Bun}_G} \mathcal{V}$ is representable by a quasi-compact scheme which is locally of finite presentation over S.*

In the proofs that follow, we will abuse notation and use i_n and π_n to denote the corresponding morphisms under a change of base $T \to S$.

The following proposition shows that Theorem 5.0.9 is useful in providing a smooth presentation of a quasi-compact open substack of Bun_G when G is smooth.

Proposition 5.0.10. *The projection $\mathrm{Bun}_G^{(nx)} \to \mathrm{Bun}_G$ sending $(\mathcal{E}, \phi) \mapsto \mathcal{E}$ is schematic, surjective, affine, and of finite presentation. If G is smooth over k, then the projection is also smooth.*

Proof. For an S-scheme T, let $\mathcal{P} \in \mathrm{Bun}_G(T)$. Then the fibered product on $(\mathbf{Sch}_{/T})^{\mathrm{op}} \to \mathbf{Gpd}$ is given by

$$\left(\mathrm{Bun}_G^{(nx)} \underset{\mathrm{Bun}_G}{\times} T\right)(T') = \left\{(\mathcal{E}, \phi, \gamma) \mid (\mathcal{E}, \phi) \in \mathrm{Bun}_G^{(nx)}(T'), \gamma : \mathcal{E} \to \mathcal{P}_{T'}\right\}$$

where a morphism $(\mathcal{E}, \phi, \gamma) \to (\mathcal{E}', \phi', \gamma')$ is a G-equivariant morphism $f : \mathcal{E} \to \mathcal{E}'$ such that $i_n^*(f) \circ \phi = \phi'$ and $\gamma' \circ f = \gamma$. We observe that the above groupoids are equivalence relations, and $\mathrm{Bun}_r^{(nx)} \times_{\mathrm{Bun}_r} T$ is isomorphic to the functor $F : (\mathbf{Sch}_{/T})^{\mathrm{op}} \to \mathbf{Set}$ sending

$$(T' \to T) \mapsto \mathrm{Hom}_{BG((nx)_{T'})}\left((nx)_{T'} \times G, i_n^*(\mathcal{P}_{T'})\right).$$

There is a canonical simply transitive right action of $\pi_{n*}G_{(nx)_T}$ on F over T (here the direct image π_{n*} of a functor is defined as in [BLR90, 7.6]). Since $G_{(nx)_T}$ is affine and finitely presented over $(nx)_T$, [BLR90, 7.6, Theorem 4, Proposition 5] show that $\pi_{n*}G_{(nx)_T}$ is representable by a group scheme which is affine and finitely presented over T. If G is smooth over k, then $\pi_{n*}G_{(nx)_T}$ is also smooth over T. Let $\left(T_i' \to (nx)_T\right)$ be an fppf cover trivializing $i_n^*\mathcal{P}$. Then the refinement $\left(T_i' \times_T (nx)_T \to (nx)_T\right)$ also trivializes $i_n^*\mathcal{P}$. Since $(nx)_T \to T$ is fppf, the compositions $\left(T_i' \to (nx)_T \to T\right)$ give an fppf cover trivializing F, which shows that F is a $\pi_{n*}G_{(nx)_T}$-torsor. We conclude that $F \to T$ has the desired properties by descent theory. $\qquad\square$

Remark 5.0.11. Proposition 5.0.10 shows that if Bun_G is an algebraic stack, then $\mathrm{Bun}_G^{(nx)}$ is also algebraic by [Sta, Lemma 05UM].

For positive integers $m < n$, we have the nilpotent thickening $i_{m,n} : (mx) \hookrightarrow (nx)$.

Proposition 5.0.12. *Suppose that G is smooth over k. For positive integers $m < n$, the morphism $\mathrm{Bun}_G^{(nx)} \to \mathrm{Bun}_G^{(mx)}$ sending $(\mathcal{P}, \phi) \mapsto (\mathcal{P}, i_{m,n}^*(\phi))$ is schematic, surjective, affine, and of finite presentation.*

Proof. For an S-scheme T, let $(\mathcal{P}, \psi) \in \mathrm{Bun}_G^{(mx)}(T)$. By similar considerations as in the proof of Proposition 5.0.10, we see that the fibered product $\mathrm{Bun}_G^{(nx)} \times_{\mathrm{Bun}_G^{(mx)}} T$ is isomorphic to the functor $F : (\mathbf{Sch}_{/T})^{\mathrm{op}} \to \mathbf{Set}$ sending

$$(T' \to T) \mapsto \left\{\phi : (nx)_{T'} \times G \to \mathcal{P}_{T'} \mid i_{m,n}^*(\phi) = \psi_{T'}\right\}.$$

The pullback $i_{m,n}^*$ defines a natural morphism

$$\pi_{n*}G_{(nx)_T} \to \pi_{m*}G_{(mx)_T}.$$

Let $N_{m,n}$ denote the kernel of this morphism. By [BLR90, 7.6, Theorem 4, Proposition 5], the Weil restrictions are representable by group schemes affine and of finite presentation over T. Then $N_{m,n}$ is a finitely presented closed subgroup of $\pi_{n*}G_{(nx)_T}$ (the identity section of $\pi_{m*}G_{(mx)_T}$ is a finitely presented closed immersion). Hence $N_{m,n}$ is representable by a scheme affine and of finite presentation over T. A T'-point of $N_{m,n}$ is an element $g \in G((nx)_{T'})$ such that $i_{m,n}^*(g) = 1$. From this description, it is evident that composition defines a canonical simply transitive right $N_{m,n}$-action on F. We show that F is an $N_{m,n}$-torsor.

From the proof of Proposition 5.0.10, we can choose an fppf covering $(T_i \to T)$ such that $i_n^*\mathcal{P}$ admits trivializations $\gamma_i : (nx)_{T_i} \times G \simeq i_n^*\mathcal{P}_{T_i}$. Since $(mx)_T, (nx)_T$ are finite over T, we may additionally assume that $(mx)_{T_i} \hookrightarrow (nx)_{T_i}$ is a nilpotent thickening of affine schemes. Since G is smooth, by the infinitesimal lifting property for smooth morphisms [BLR90, 2.2, Proposition 6], we have a surjection

$$G((nx)_{T_i}) \twoheadrightarrow G((mx)_{T_i}).$$

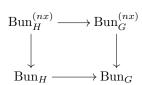Therefore there exists automorphisms ϕ_i of $(nx)_{T_i} \times G$ such that

$$i^*_{m,n}(\phi_i) = i^*_{m,n}(\gamma_i^{-1}) \circ \psi_{T_i}.$$

We have shown that $\gamma_i \circ \phi_i \in F(U_i) \neq \emptyset$. It follows that F is an $N_{m,n}$-torsor, and $F \to T$ has the desired properties by descent theory. $\qquad\square$

Once again, our plan for proving Theorem 5.0.9 is to reduce to considering GL_r-bundles. The following lemma makes this possible.

Lemma 5.0.13. *Let $H \hookrightarrow G$ be a closed subgroup of G. There is a finitely presented closed immersion* $\mathrm{Bun}_H^{(nx)} \hookrightarrow \mathrm{Bun}_G^{(nx)} \times_{\mathrm{Bun}_G} \mathrm{Bun}_H$.

Proof. The change of group morphism $_{(-)}G : BH \to BG$ of Lemma 2.4.1 induces a 2-commutative square

$$\begin{array}{ccc} \mathrm{Bun}_H^{(nx)} & \longrightarrow & \mathrm{Bun}_G^{(nx)} \\ \downarrow & & \downarrow \\ \mathrm{Bun}_H & \longrightarrow & \mathrm{Bun}_G \end{array}$$

The 2-fibered product $\mathrm{Bun}_G^{(nx)} \times_{\mathrm{Bun}_G} \mathrm{Bun}_H$ is isomorphic to the stack $\mathcal{F} : (\mathbf{Sch}_{/S})^{\mathrm{op}} \to \mathbf{Gpd}$ defined by

$$\mathcal{F}(T) = \left\{ (\mathcal{P}, \phi) \mid \mathcal{P} \in \mathrm{Bun}_H(T),\, \phi : (nx)_T \times G \to {}_{(i_n^* \mathcal{P})}G \right\},$$

where a morphism $(\mathcal{P}, \phi) \to (\mathcal{P}, \phi')$ is an H-equivariant morphism $f : \mathcal{P} \to \mathcal{P}'$ satisfying $_{(i_n^* f)}G \circ \phi = \phi'$. We show that the morphism $\mathrm{Bun}_H^{(nx)} \to \mathcal{F}$ induced by the 2-commutative square is a finitely presented closed immersion. Fix an S-scheme T and take $(\mathcal{P}, \phi) \in \mathcal{F}(T)$. We have a 2-commutative square

$$\begin{array}{ccc} (nx)_T & =\!=\!= & (nx)_T \\ {\scriptstyle i_n^* \mathcal{P}}\downarrow & & \downarrow{\scriptstyle (nx)_T \times G} \\ BH & \longrightarrow & BG \end{array}$$

via ϕ. From the proof of Lemma 2.4.1, we know that $(nx)_T \times_{BG} BH \simeq (nx)_T \times (H\backslash G)$. This implies that

$$(nx)_T \underset{(i_n^*\mathcal{P}, \phi),\, (nx)_T \underset{BG}{\times} BH,\, ((nx)_T \times H, \mathrm{id})}{\times} (nx)_T$$

is representable by a finitely presented closed subscheme $T_0 \subset (nx)_T$. From the definition of the 2-fibered product, we have that $\mathrm{Hom}(T', T_0)$ consists of the morphisms $u : T' \to (nx)_T$ such that there exists a $\psi : T' \times H \to u^* i_n^* \mathcal{P}$ with $_{(\psi)}G = u^*(\phi)$. If such a ψ exists, then it must be unique by definition of the 2-fibered product and the fact that T_0 is a scheme. From this description, we observe that

$$\mathrm{Bun}_H^{(nx)} \underset{\mathcal{F}}{\times} T \simeq \pi_{n*} T_0.$$

Then [BLR90, 7.6, Propositions 2, 5] imply that $\pi_{n*} T_0$ is a finitely presented closed subscheme of $\pi_{n*}(nx)_T \simeq T$. $\qquad\square$

Lemma 4.1.1 implies that $\mathrm{Bun}_{\mathrm{GL}_r}^{(nx)} \simeq \mathrm{Bun}_r^{(nx)}$, where $\mathrm{Bun}_r^{(nx)}(T)$ is the groupoid of pairs $(\mathcal{E}, \phi)$ for $\mathcal{E}$ a locally free $\mathcal{O}_{X_T}$-module of rank r and an isomorphism

$$\phi : \mathcal{O}_{(nx)_T}^r \simeq i_n^* \mathcal{E}$$

of $\mathcal{O}_{(nx)_T}$-modules. We show that the projections $\mathrm{Bun}_r^{(nx)} \to \mathrm{Bun}_r$ give smooth presentations of the open substacks $\mathcal{U}_n^\Phi \hookrightarrow \mathrm{Bun}_r$ defined in §4.

Theorem 5.0.14. *Suppose $p : X \to S$ is a flat, strongly projective morphism with geometrically integral fibers over a quasi-compact base scheme S. For any integer n and polynomial $\Phi \in \mathbf{Q}[\lambda]$, there exists an integer N_0 such that for all $N \geq N_0$, the 2-fibered product $\mathrm{Bun}_r^{(Nx)} \times_{\mathrm{Bun}_r} \mathcal{U}_n^\Phi$ is representable by a finitely presented, quasi-projective S-scheme.*

Given Theorem 5.0.14, let us deduce Theorem 5.0.9.

Proof of Theorem 5.0.9. Embed G into some GL_r. Take an open covering $(S_i \subset S)_{i \in I}$ by affine subschemes such that the restrictions $X_{S_i} \to S_i$ are strongly projective. Let $J' = I \times \mathbf{Z} \times \mathbf{Q}[\lambda]$, and for $j = (i, n, \Phi) \in J'$, denote $\mathcal{U}_j = \mathcal{U}_n^\Phi \times_S S_i$. Theorem 5.0.14 implies that for any $j \in J'$, there exists an integer $N_{j,0}$ such that for all $N \geq N_{j,0}$, the 2-fibered product $\mathrm{Bun}_r^{(Nx)} \times_{\mathrm{Bun}_r} \mathcal{U}_j$ is representable by a scheme. Let $\widetilde{\mathcal{U}}_j = \mathrm{Bun}_G \times_{\mathrm{Bun}_r} \mathcal{U}_j$. By quasi-compactness of $\mathcal{V}$, there exists a finite subset $J \subset J'$ such that

$$\mathcal{V} \subset \bigcup_{j \in J} \widetilde{\mathcal{U}}_j$$

is an open immersion [Sta, Lemmas 05UQ, 05UR]. Letting N_0 equal the maximum of the $N_{j,0}$ over all $j \in J$, we have that for all $N \geq N_0$, $j \in J$, the 2-fibered product $\mathrm{Bun}_r^{(Nx)} \times_{\mathrm{Bun}_r} \mathcal{U}_j$ is representable by an scheme. Corollary 3.2.4 implies that $\mathrm{Bun}_G \to \mathrm{Bun}_r$ is schematic, and Lemma 5.0.13 shows that $\mathrm{Bun}_G^{(Nx)} \to \mathrm{Bun}_r^{(Nx)} \times_{\mathrm{Bun}_r} \mathrm{Bun}_G$ is a closed immersion. Therefore by base change we deduce that

$$\mathrm{Bun}_G^{(Nx)} \underset{\mathrm{Bun}_G}{\times} \widetilde{\mathcal{U}}_j$$

is representable by a scheme. Now it follows from [Sta, Lemma 05WF] that $\mathrm{Bun}_G^{(Nx)} \times_{\mathrm{Bun}_G} \mathcal{V}$ is representable by an S-scheme.

From Theorem 1.0.1 we know that $\mathcal{V}$ is a quasi-compact algebraic stack locally of finite presentation over S. Proposition 5.0.10 implies that $\mathrm{Bun}_G^{(Nx)} \to \mathrm{Bun}_G$ is schematic and finitely presented, so we deduce that $\mathrm{Bun}_G^{(Nx)} \times_{\mathrm{Bun}_G} \mathcal{V}$ is represented by a quasi-compact scheme locally of finite presentation over S. $\square$

The rest of this section is devoted to proving Theorem 5.0.14.

Lemma 5.0.15. *Suppose S is the spectrum of a field k', and X is integral and proper. For a locally free $\mathcal{O}_X$-module $\mathcal{F}$ of finite rank, there exists an integer N_0 such that for all integers $N \geq N_0$, the unit morphism*

$$\eta_N : \Gamma(X, \mathcal{F}) \to \Gamma(X, i_{N*} i_N^* \mathcal{F}) \simeq \Gamma(X, \mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_X / \mathcal{I}_x^N)$$

is injective.

Proof. Let N be an arbitrary positive integer. Since $\mathcal{F}$ is X-flat, we have a short exact sequence

$$0 \to \mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{I}_x^N \to \mathcal{F} \to \mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_X / \mathcal{I}_x^N \to 0.$$

Thus the kernel of η_N is $\Gamma(X, \mathcal{F} \otimes \mathcal{I}_x^N)$. Since X is proper, $\Gamma(X, \mathcal{F} \otimes \mathcal{I}_x^N)$ is a finite dimensional k'-vector space for each N, and we have a descending chain

$$\cdots \supset \Gamma(X, \mathcal{F} \otimes \mathcal{I}_x^N) \supset \Gamma(X, \mathcal{F} \otimes \mathcal{I}_x^{N+1}) \supset \cdots$$

and this chain must stabilize. Suppose the chain does not stabilize to 0. Then there is a nonzero global section $f \in \Gamma(X, \mathcal{F})$ lying in $\Gamma(X, \mathcal{F} \otimes \mathcal{I}_x^N)$ for all N. Thus the image of f in

$$\Gamma(X, \mathcal{F}) \to \mathcal{F}_x \to \widehat{\mathcal{F}}_x$$

is 0. Since X is assume integral, $\Gamma(X, \mathcal{F}) \hookrightarrow \mathcal{F}_x$ is injective. By Krull's intersection theorem [Mat80, 11.D, Corollary 3], the morphism $\mathcal{F}_x \hookrightarrow \widehat{\mathcal{F}}_x$ is injective. Therefore $f = 0$, a contradiction. We conclude that there exists N_0 such that for all $N \geq N_0$ for some N_0, the kernel $\ker \eta_N = \Gamma(X, \mathcal{F} \otimes \mathcal{I}_x^N) = 0$. $\qquad\square$

Lemma 5.0.16. *Suppose $p : X \to S$ is a flat, strongly projective morphism with geometrically integral fibers. For a quasi-compact S-scheme T, let $\mathcal{E} \in \mathcal{U}_n(T)$. Then there exists an integer N_0 such that for all integers $N \geq N_0$, the dual of the unit morphism*

$$(5.0.16.1) \qquad \left(p_{T*} i_{N*} i_N^* (\mathcal{E}(n))\right)^\vee \to \left(p_{T*}(\mathcal{E}(n))\right)^\vee$$

is a surjective morphism of locally free $\mathcal{O}_T$-modules.

Proof. Let $\mathcal{F} = \mathcal{E}(n)$. From Proposition 4.1.3, we know that $p_{T*}\mathcal{F}$ is flat, and $\mathcal{F}$ is cohomologically flat over T in all degrees. Let N be an arbitrary positive integer. By the projection formula, $i_{N*} i_N^* \mathcal{F} \simeq \mathcal{F} \otimes_{\mathcal{O}_{X_T}} i_{N*} \mathcal{O}_{(Nx)_T}$. Since $\pi_N : (Nx)_T \to T$ is finite and locally free, $i_{N*} \mathcal{O}_{(Nx)_T}$ is T-flat. We deduce that $i_{N*} i_N^* \mathcal{F}$ is T-flat because $\mathcal{F}$ is locally free on X_T. Additionally, $p_{T*} i_{N*} i_N^* \mathcal{F} = \pi_{N*}(i_N^* \mathcal{F})$ is locally free on T because $i_N^* \mathcal{F}$ and π_N are locally free. By the Leray spectral sequence, we have quasi-isomorphisms

$$Rp_{T*} i_{N*} i_N^* \mathcal{F} \simeq R\pi_{N*} i_N^* \mathcal{F} \simeq \pi_{N*} i_N^* \mathcal{F}$$

in the derived category $D(T)$ of $\mathcal{O}_T$-modules, since π_N is affine. Now Lemma 4.1.2 implies $i_{N*} i_N^* \mathcal{F}$ is cohomologically flat over T in all degrees. Remark 4.1.7 implies $p_{T*}\mathcal{F}$ is locally free, so $p_{T*}\mathcal{F} \to p_{T*} i_{N*} i_N^* \mathcal{F}$ is a morphism of locally free $\mathcal{O}_T$-modules. Take a point $t \in T$. By Lemma 5.0.15, there exists an integer N_t such that for all $N \geq N_t$, the unit morphism $\Gamma(X_t, \mathcal{F}_t) \to \Gamma(X_t, i_{N*} i_N^* \mathcal{F}_t)$ of finite dimensional $\kappa(t)$-vector spaces is injective. By cohomological flatness, this implies

$$(p_{T*}\mathcal{F}) \underset{\mathcal{O}_T}{\otimes} \kappa(t) \to (p_{T*} i_{N*} i_N^* \mathcal{F}) \underset{\mathcal{O}_T}{\otimes} \kappa(t)$$

is injective (observe that $i_{N*} i_N^*$ commutes with base change). Since taking duals commutes with base change for locally free modules, we have that $(p_{T*} i_{N*} i_N^* \mathcal{F})^\vee \otimes \kappa(t) \to (p_{T*}\mathcal{F})^\vee \otimes \kappa(t)$ is surjective. By Nakayama's lemma, there exists an open subscheme $U_t \subset T$ containing t on which the restriction $(p_{T*} i_{N*} i_N^* \mathcal{F})^\vee|_{U_t} \to (p_{T*}\mathcal{F})^\vee|_{U_t}$ is surjective. By quasi-compactness, we can cover T by finitely many U_t. Taking N_0 to be the maximum of the corresponding N_t proves the claim. $\qquad\square$

Lemma 5.0.17. *Suppose $p : X \to S$ is a flat, strongly projective morphism with geometrically integral fibers over a quasi-compact base scheme S. For any integer n and polynomial $\Phi \in \mathbf{Q}[\lambda]$, there exists an integer N_0 such that for all $N \geq N_0$, the morphism (5.0.16.1) is a surjective morphism of locally free $\mathcal{O}_T$-modules for any S-scheme T and $\mathcal{E} \in \mathcal{U}_n^\Phi(T)$.*

Proof. From Theorem 1.0.1 we get a quasi-compact scheme Y_n^Φ and a schematic, smooth, surjective morphism $Y_n^\Phi \to \mathcal{U}_n^\Phi$. By the 2-Yoneda lemma, the previous morphism corresponds to a locally free sheaf $\mathcal{E}_0 \in \mathcal{U}_n^\Phi(Y_n^\Phi)$. By Lemma 5.0.16, there exists N_0 such that for all $N \geq N_0$,

the morphism (5.0.16.1) is surjective for $\mathcal{E}_0$. Now for any S-scheme T and $\mathcal{E} \in \mathcal{U}_n^\Phi(T)$, let

$$
\begin{array}{ccc}
T' & \longrightarrow & T \\
\downarrow & & \downarrow{\scriptstyle \mathcal{E}} \\
Y_n^\Phi & \xrightarrow{\mathcal{E}_0} & \mathcal{U}_n^\Phi
\end{array}
$$

be 2-Cartesian, where T' is a scheme and $T' \to T$ is smooth and surjective (and hence faithfully flat). Thus there is an isomorphism $(\mathcal{E}_0)_{T'} \simeq \mathcal{E}_{T'}$. From the proof of Lemma 5.0.16, we know that $i_{N*}i_N^*\mathcal{E}_0(n)$ and $\mathcal{E}_0(n)$ are cohomologically flat over Y_n^Φ, and the analogous assertion is true for $\mathcal{E}$ over T. Therefore pulling back along $T' \to Y_n^\Phi$ implies that (5.0.16.1) is surjective for $\mathcal{E}_{T'} \in \mathcal{U}_n^\Phi(T')$. Since $T' \to T$ is faithfully flat, we deduce that (5.0.16.1) is surjective for $\mathcal{E} \in \mathcal{U}_n^\Phi(T)$. $\qquad\square$

Proof of Theorem 5.0.14. Let N_0 be an integer satisfying the assertions of Lemma 5.0.17. Fix an integer $N \geq N_0$, and consider the coherent sheaf $\mathcal{M} = \left(\pi_{N*}i_N^*(\mathcal{O}_X^r(n))\right)^\vee$, which is locally free of finite rank since $i_N^*(\mathcal{O}_X^r(n))$ and π_N are. By [EGAG, Proposition 9.7.7, 9.8.4], the Grassmannian functor $\mathrm{Grass}(\mathcal{M}, \Phi(n))$ is representable by a strongly projective S-scheme. We have a morphism

$$\mathrm{Grass}(\mathcal{M}, \Phi(n)) \to B\mathrm{GL}_{\Phi(n)} \times S$$

sending a T-point $\mathcal{M}_T \twoheadrightarrow \mathcal{F}$ to $\mathcal{F}^\vee$. From Lemmas 2.1.1 and 4.1.11, we have an isomorphism $\mathcal{U}_n^\Phi \simeq [Y_n^\Phi/\mathrm{GL}_{\Phi(n)}]$ which sends $\mathcal{E} \in \mathcal{U}_n^\Phi(T)$ to $\mathrm{Isom}_T\left(\mathcal{O}_T^{\Phi(n)}, p_{T*}(\mathcal{E}(n))\right) \to Y_n^\Phi$, where the $\mathrm{GL}_{\Phi(n)}$-bundle corresponds to the locally free $\mathcal{O}_T$-module $p_{T*}(\mathcal{E}(n))$. Therefore the change of space morphism $[Y_n^\Phi/\mathrm{GL}_{\Phi(n)}] \to [S/\mathrm{GL}_{\Phi(n)}]$ is isomorphic to the morphism $\mathcal{U}_n^\Phi \to B\mathrm{GL}_{\Phi(n)} \times S$ sending $\mathcal{E} \mapsto p_{T*}(\mathcal{E}(n))$ by Remark 2.3.5. By Lemmas 2.3.1 and 4.1.12, we deduce that the morphism $\mathcal{U}_n^\Phi \to B\mathrm{GL}_{\Phi(n)} \times S$ is schematic, quasi-projective, and finitely presented. We have a Cartesian square

$$
\begin{array}{ccc}
F_0 & \longrightarrow & \mathrm{Grass}(\mathcal{M}, \Phi(n)) \\
\downarrow & & \downarrow \\
\mathcal{U}_n^\Phi & \longrightarrow & B\mathrm{GL}_{\Phi(n)} \times S
\end{array}
$$

where F_0 is a functor $(\mathbf{Sch}_{/S})^{\mathrm{op}} \to \mathbf{Set}$ sending

$$(T \to S) \mapsto \left\{(\mathcal{E}, q) \mid \mathcal{E} \in \mathcal{U}_n^\Phi(T), q : \mathcal{M} \twoheadrightarrow (p_{T*}(\mathcal{E}(n)))^\vee\right\}/\sim$$

where $(\mathcal{E}, q) \sim (\mathcal{E}', q')$ if there exists (a necessarily unique) isomorphism $f : \mathcal{E} \simeq \mathcal{E}'$ such that $(p_{T*}(f_n))^\vee \circ q' = q$. Note that F_0 is representable by a scheme finitely presented and quasi-projective over $\mathrm{Grass}(\mathcal{M}, \Phi(n))$. By [EGAII, Proposition 5.3.4(ii)], we find that F_0 is finitely presented and quasi-projective over S.

For an S-scheme T, let $(\mathcal{E}, \phi), (\mathcal{E}', \phi') \in \mathrm{Bun}_r^{(Nx)}(T)$ such that $\mathcal{E}, \mathcal{E}' \in \mathcal{U}_n^\Phi(T)$. Suppose we have a morphism $f : (\mathcal{E}, \phi) \to (\mathcal{E}', \phi')$ in $\mathrm{Bun}_r^{(Nx)}$. Then Lemma 5.0.16 gives us a commutative diagram

$$
\begin{array}{ccc}
\mathcal{M}_T \xrightarrow[\sim]{(\pi_{N*}(\phi_n'^{-1}))^\vee} \left(p_{T*}i_{N*}i_N^*(\mathcal{E}'(n))\right)^\vee & \longrightarrow\!\!\!\!\!\!\rightarrow & \left(p_{T*}(\mathcal{E}'(n))\right)^\vee \\
\| \qquad\qquad\qquad \downarrow{\scriptstyle (p_{T*}i_{N*}i_N^*(f_n))^\vee} & & \downarrow{\scriptstyle (p_{T*}(f_n))^\vee} \\
\mathcal{M}_T \xrightarrow[\sim]{(\pi_{N*}(\phi_n^{-1}))^\vee} \left(p_{T*}i_{N*}i_N^*(\mathcal{E}(n))\right)^\vee & \longrightarrow\!\!\!\!\!\!\rightarrow & \left(p_{T*}(\mathcal{E}(n))\right)^\vee
\end{array}
$$

where the horizontal arrows q', q are surjections. Observe that f gives an equivalence $(\mathcal{E}, q) \sim (\mathcal{E}', q')$ in $F_0(T)$, and consequently f must be unique. Therefore $\mathrm{Bun}_r^{(Nx)} \times_{\mathrm{Bun}_r} \mathcal{U}_n^\Phi$ is isomorphic to the functor $F : (\mathbf{Sch}_{/S})^{\mathrm{op}} \to \mathbf{Set}$ sending

$$(T \to S) \mapsto \{(\mathcal{E}, \phi) \mid \mathcal{E} \in \mathcal{U}_n^\Phi(T), \phi : \mathcal{O}_{(Nx)_T}^r \simeq i_N^* \mathcal{E}\}/\sim.$$

By the previous discussion, we have a morphism $F \to F_0$ by sending $(\mathcal{E}, \phi) \mapsto (\mathcal{E}, q)$. We show that this morphism is a finitely presented, locally closed immersion, which will then imply that F is representable by a finitely presented, quasi-projective S-scheme.

The claim is Zariski local on S, so we reduce to the case where S is noetherian by Remark 4.1.4. Let T be an S-scheme and $(\mathcal{E}, q) \in F_0(T)$. Then $\mathcal{R} = \pi_{N*}\mathcal{O}_{(Nx)_T}$ is a coherent $\mathcal{O}_T$-algebra on T, and since π_N is affine, [EGAII, Proposition 1.4.3] implies that

$$\pi_{N*} : \mathrm{QCoh}((Nx)_T) \to \mathrm{QCoh}(T)$$

is a faithful embedding to the subcategory of quasi-coherent $\mathcal{O}_T$-modules with structures of $\mathcal{R}$-modules, and morphisms are those of $\mathcal{R}$-modules. Since $\mathcal{E}(n)$ is relatively generated by global sections, the counit $p_T^* p_{T*}(\mathcal{E}(n)) \to \mathcal{E}(n)$ is surjective. Applying i_N^* gives a surjection $\pi_N^* p_{T*}(\mathcal{E}(n)) \to i_N^*(\mathcal{E}(n))$. As π_{N*} is exact on quasi-coherent sheaves, we have a surjection

$$\mathrm{id}_\mathcal{R} \overline{\otimes} \eta : \mathcal{R} \underset{\mathcal{O}_T}{\otimes} p_{T*}(\mathcal{E}(n)) \simeq \pi_{N*}\pi_N^* p_{T*}(\mathcal{E}(n)) \twoheadrightarrow \pi_{N*} i_N^*(\mathcal{E}(n)),$$

and this is the morphism of $\mathcal{R}$-modules induced by the unit $\eta : p_{T*}(\mathcal{E}(n)) \to p_{T*} i_{N*} i_N^*(\mathcal{E}(n))$. Now the dual morphism $q^\vee : p_{T*}(\mathcal{E}(n)) \to \mathcal{M}_T^\vee$ also induces a morphism of $\mathcal{R}$-modules

$$\mathrm{id}_\mathcal{R} \overline{\otimes} q^\vee : \mathcal{R} \underset{\mathcal{O}_T}{\otimes} p_{T*}(\mathcal{E}(n)) \to \mathcal{M}_T^\vee \simeq \pi_{N*} i_N^*(\mathcal{O}_X^r(n)).$$

If $(\mathcal{E}, q)$ is the image of some $(\mathcal{E}, \phi) \in F(T)$, then the diagram

(5.0.17.1)

$$
\begin{array}{ccc}
& \mathcal{R} \otimes p_{T*}(\mathcal{E}(n)) & \\
{\scriptstyle \mathrm{id}_\mathcal{R} \overline{\otimes} q^\vee} \swarrow & & \searrow {\scriptstyle \mathrm{id}_\mathcal{R} \overline{\otimes} \eta} \\
\pi_{N*} i_N^*(\mathcal{O}_X^r(n)) \underset{\sim}{\xrightarrow{\ \pi_{N*}\phi_n\ }} & & \pi_{N*} i_N^*(\mathcal{E}(n))
\end{array}
$$

is commutative. In this case, $\mathrm{id}_\mathcal{R} \overline{\otimes} q^\vee$ is surjective. Since π_{N*} is faithful, (5.0.17.1) implies that if such a ϕ exists, then it is unique. Conversely, if $\mathrm{id}_\mathcal{R} \overline{\otimes} q^\vee$ is surjective and a factorization as in (5.0.17.1) exists, then such a ϕ does exist. Indeed, any morphism

$$\pi_{N*} i_N^*(\mathcal{O}_X^r(n)) \to \pi_{N*} i_N^*(\mathcal{E}(n))$$

making the triangle commute is necessarily a surjective morphism of $\mathcal{R}$-modules. Therefore it is in the image of π_{N*}. Tensoring by $i_N^*(\mathcal{O}(-n))$ gives the desired ϕ, which is an isomorphism since it is a surjection of locally free $\mathcal{O}_{(Nx)_T}$-modules of rank r. Define the open subscheme $U \subset T$ to be the complement of the support of $\mathrm{coker}(\mathrm{id}_\mathcal{R} \overline{\otimes} q^\vee)$. By cohomological flatness (see Lemma 5.0.16), the morphisms $\eta, q^\vee$ commute with pullback along a morphism $T' \to T$. We also have $\mathcal{R}_{T'} \simeq \pi_{N*}\mathcal{O}_{(Nx)_{T'}}$ since π_N is affine. Therefore $\mathrm{id}_\mathcal{R} \overline{\otimes} \eta$, $\mathrm{id}_\mathcal{R} \overline{\otimes} q^\vee$ commute with pullback along $T' \to T$. By Nakayama's lemma and faithful flatness of field extensions, we deduce that $\mathrm{id}_{\mathcal{R}_{T'}} \overline{\otimes} q_{T'}^\vee$ is surjective if and only if $T' \to T$ lands in U. Let

$$\mathcal{K} = \ker(\mathrm{id}_\mathcal{R} \overline{\otimes} q^\vee|_U),$$

which is a coherent $\mathcal{O}_U$-module. By [EGAG, Lemme 9.7.9.1], there exists a closed subscheme $V \subset U$ with the universal property that a morphism $u : T' \to U$ factors through V if and only if $u^*(\mathrm{id}_\mathcal{R} \overline{\otimes} \eta)(\mathcal{K}_{T'}) = 0$. Since $\mathrm{id}_\mathcal{R} \overline{\otimes} q^\vee|_U$ is surjective and $\pi_{N*} i_N^*(\mathcal{O}_X^r(n))|_U$ is U-flat, we have $\mathcal{K}_{T'} \simeq \ker(u^*(\mathrm{id}_\mathcal{R} \overline{\otimes} q^\vee))$. Therefore u factors through V if and only if the diagram (5.0.17.1)

admits a factorization on T'. We conclude that V represents the fibered product $F \times_{F_0} T$. Taking $T = F_0$, which is noetherian, we see that $F \to F_0$ is a finitely presented immersion. $\square$

6. SMOOTHNESS

We have already seen that Bun_G is locally of finite presentation. We now show that Bun_G is smooth over S when G is smooth and $X \to S$ is a relative curve.

Proposition 6.0.18. *Suppose G is smooth over k and $p : X \to S$ is a flat, finitely presented, projective morphism of k-schemes with fibers of dimension 1. Then Bun_G is smooth over S.*

We remind the reader that, following [LMB00, Définition 4.14], an algebraic S-stack $\mathcal{X}$ is smooth over S if there exists a scheme U smooth over S and a smooth surjective morphism $U \to \mathcal{X}$.
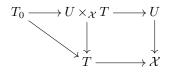
6.1. **Infinitesimal lifting criterion.** We first provide an infinitesimal lifting criterion for smoothness of an algebraic S-stack. The next lemma in fact follows from the proofs of [BLR90, 2.2, Proposition 6] and [LMB00, Proposition 4.15], but we reproduce the proof to emphasize that we can impose a noetherian assumption.

Lemma 6.1.1. *Let S be a locally noetherian scheme and $\mathcal{X}$ an algebraic stack locally of finite type over S with a schematic diagonal. Suppose that for any affine noetherian scheme T and a closed subscheme T_0 defined by a square zero ideal, given a 2-commutative square*

$$
\begin{array}{ccc}
T_0 & \longrightarrow & \mathcal{X} \\
\downarrow & \nearrow & \downarrow \\
T & \longrightarrow & S
\end{array}
$$

of solid arrows, there exists a lift $T \to \mathcal{X}$ making the diagram 2-commutative. Then $\mathcal{X} \to S$ is smooth.

Proof. Let U be a scheme locally of finite type over S with a smooth surjective morphism to $\mathcal{X}$. We want to show $U \to S$ is smooth. This is local on S, so we may assume $S = \mathrm{Spec}\, R$ is an affine noetherian scheme. Suppose we have a morphism $T_0 \to U$ over S. Then the hypothesis allows us to find a lift $T \to \mathcal{X}$. We have a 2-commutative diagram

$$
\begin{array}{ccccc}
T_0 & \longrightarrow & U \times_{\mathcal{X}} T & \longrightarrow & U \\
 & \searrow & \downarrow & & \downarrow \\
 & & T & \longrightarrow & \mathcal{X}
\end{array}
$$

where $U \times_{\mathcal{X}} T \to T$ is isomorphic to a smooth morphism of scheme. By [BLR90, 2.2, Proposition 6], there exists a section $T \to U \times_{\mathcal{X}} T$. This shows that

$$(6.1.1.1) \qquad\qquad\qquad \mathrm{Hom}_S(T, U) \twoheadrightarrow \mathrm{Hom}_S(T_0, U)$$

is surjective. Note that (6.1.1.1) still holds if we replace U by an open subscheme, since T_0 and T have the same underlying topological space. Since smoothness is local on the source, we can assume $U = \mathrm{Spec}\, B$ is affine. Then $U \to S$ of finite type implies that U is a closed subscheme of $\mathbf{A}^n_S$. Let $A = R[t_1, \ldots, t_n]$ and $I \subset A$ the ideal corresponding to $U \hookrightarrow \mathbf{A}^n_S$. By [SGA1, II, Théorème 4.10], it suffices to show that the canonical sequence

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega_{\mathbf{A}^n_S/S} \otimes \mathcal{O}_U \to \Omega_{U/S} \to 0$$

is locally split exact (we know *a priori* that the sequence is right exact). Note that A is noetherian, and $\mathcal{I}/\mathcal{I}^2$ is the coherent sheaf corresponding to I/I^2. Since A/I^2 is noetherian, by (6.1.1.1) the R-morphism

$$\mathrm{id} : A/I \to A/I = (A/I^2)/(I/I^2)$$

lifts to a morphism $\varphi : A/I \to A/I^2$ of R-algebras. This implies that the short exact sequence

$$0 \to I/I^2 \overset{\iota}{\to} A/I^2 \overset{\pi}{\to} A/I \to 0$$

splits since $\pi \circ \varphi = \mathrm{id}_{A/I}$. Let $\delta = \mathrm{id}_{A/I^2} - \varphi \circ \pi : A/I^2 \to I/I^2$ be a left inverse to ι. Since $\delta(a)\delta(b) = 0$ for all $a, b \in A/I^2$, we have

$$\delta(ab) = ab - (\varphi \circ \pi)(ab) + (a - (\varphi \circ \pi)(a))(b - (\varphi \circ \pi)(b)) = a\delta(b) + b\delta(a)$$

using the fact that $\varphi \circ \pi$ is a morphism of R-algebras. Thus δ gives an R-derivation $A \to I/I^2$, which corresponds to a morphism $\Omega_{A/R} \to I/I^2$ of A-modules. Since $\delta \circ \iota = \mathrm{id}_{I/I^2}$, this morphism defines a left inverse $\Omega_{A/R} \otimes_A B \to I/I^2$, which shows that

$$0 \to I/I^2 \to \Omega_{A/R} \underset{A}{\otimes} B \to \Omega_{B/R} \to 0$$

is split exact. We conclude that $U \to S$ is smooth. $\qquad\square$

6.2. **Lifting gerbes.** Our aim is to prove Proposition 6.0.18 by showing that Bun_G satisfies the lifting criterion of Lemma 6.1.1. In other words, we want to lift certain torsors. To discuss when torsors lift, we will need to use the categorical language of a gerbe over a Picard stack. We refer the reader to [DG02, 3] or [Gir71] for the relevant definitions.

Let $\mathcal{C}$ be a subcanonical site and X a terminal object of $\mathcal{C}$. Suppose we have a short exact sequence of sheaves of groups

$$1 \to \mathcal{A} \to \mathcal{G} \overset{\pi}{\to} \mathcal{G}_0 \to 1$$

on $\mathcal{C}$, where $\mathcal{A}$ is abelian. Then $\mathcal{G}$ acts on $\mathcal{A}$ via conjugation, which induces a left action of $\mathcal{G}_0$ on $\mathcal{A}$, since $\mathcal{A}$ is abelian. Fix a right $\mathcal{G}_0$-torsor $\mathcal{P}$ over X. Then the twisted sheaf $_{\mathcal{P}}\mathcal{A}$ is still a sheaf of abelian groups because the $\mathcal{G}_0$-action is induced by conjugation. We define a stack $\mathcal{Q}_{\mathcal{P}} : \mathcal{C} \to \mathbf{Gpd}$ by letting

$$\mathcal{Q}_{\mathcal{P}}(U) = \left\{ (\widetilde{\mathcal{P}}, \phi) \mid \widetilde{\mathcal{P}} \in \mathrm{Tors}(\mathcal{G})(U), \, \phi : \mathcal{P}|_U \to \, _{\widetilde{\mathcal{P}}}\mathcal{G}_0 \right\}$$

where ϕ is $\mathcal{G}_0$-equivariant, and a morphism $(\widetilde{\mathcal{P}}, \phi) \to (\widetilde{\mathcal{P}}', \phi')$ in the groupoid is a $\mathcal{G}$-equivariant morphism $f : \widetilde{\mathcal{P}} \to \widetilde{\mathcal{P}}'$ such that $_f\mathcal{G}_0 \circ \phi = \phi'$.

Lemma 6.2.1. *The stack $\mathcal{Q}_{\mathcal{P}}$ of liftings of $\mathcal{P}$ to $\mathcal{G}$ is a gerbe over the Picard stack $\mathrm{Tors}(_{\mathcal{P}}\mathcal{A})$.*

Proof. We first describe the action of $\mathrm{Tors}(_{\mathcal{P}}\mathcal{A})(U)$ on $\mathcal{Q}_{\mathcal{P}}(U)$ for $U \in \mathcal{C}$. Let $\mathcal{T}$, $(\widetilde{\mathcal{P}}, \phi)$ be objects in the aforementioned categories, respectively. We can pick a covering $(U_i \to U)$ trivializing $\mathcal{P}|_U, \mathcal{T}$, and $\widetilde{\mathcal{P}}$. This in particular implies that $\mathcal{T}|_{U_i} \simeq \mathcal{A}|_{U_i}$. Denote $U_{ij} = U_i \times_U U_j$ and $U_{ijk} = U_i \times_U U_j \times_U U_k$ for indices i, j, k. Let

$$(\mathcal{G}_0|_{U_{ij}}, g_{ij})$$

be a descent datum for $\mathcal{P}|_U$, where $g_{ij} \in \mathcal{G}_0(U_{ij})$. Then $_{\mathcal{P}}\mathcal{A}|_U$ has a descent datum $(\mathcal{A}|_{U_{ij}}, \gamma_{ij})$ with $\gamma_{ij}(a) = g_{ij} a g_{ij}^{-1}$. Now since $\mathcal{T}$ is a $_{\mathcal{P}}\mathcal{A}|_U$-torsor, the transition morphisms in a descent datum for $\mathcal{T}$ are $_{\mathcal{P}}\mathcal{A}|_U$-equivariant. Thus $\mathcal{T}$ has a descent datum $(\mathcal{A}|_{U_{ij}}, \psi_{ij})$ where

$$\psi_{ij}(a) = \gamma_{ij}(a_{ij} a) = g_{ij} a_{ij} a g_{ij}^{-1}$$

for $a_{ij} \in \mathcal{A}(U_{ij})$ under the group operation of $\mathcal{G}$ (we omit restriction symbols). Now let $(\mathcal{G}|_{U_{ij}}, h_{ij})$ be a descent datum for $\widetilde{\mathcal{P}}$, where $h_{ij} \in \mathcal{G}(U_{ij})$. The cocycle condition says that $h_{ij}h_{jk} = h_{ik} \in \mathcal{G}(U_{ijk})$. Now ϕ corresponds to a morphism of descent data

$$(\phi_i) : (\mathcal{G}_0|_{U_i}, g_{ij}) \to (\mathcal{G}_0|_{U_i}, \pi(h_{ij})).$$

By possibly refining our cover further, we may assume that $\phi_i = \pi(c_i)$ for $c_i \in \mathcal{G}(U_i)$. Set $\widetilde{g}_{ij} = c_i^{-1}h_{ij}c_j \in \mathcal{G}(U_{ij})$. Now $\widetilde{g}_{ij}\widetilde{g}_{jk} = \widetilde{g}_{ik}$ and $\pi(\widetilde{g}_{ij}) = g_{ij}$. Therefore the cocycle condition $\psi_{ij} \circ \psi_{jk} = \psi_{ik}$ is equivalent to

$$(6.2.1.1) \qquad \widetilde{g}_{ij}a_{ij}\widetilde{g}_{jk}a_{jk}\widetilde{g}_{jk}^{-1}\widetilde{g}_{ij}^{-1} = \widetilde{g}_{ik}a_{ik}\widetilde{g}_{ik}^{-1} \iff h_{ij}c_ja_{ij}c_j^{-1}h_{jk}c_ka_{jk} = h_{ik}c_ka_{ik}.$$

Therefore $(\mathcal{G}|_{U_i}, h_{ij}c_ja_{ij}c_j^{-1})$ defines a descent datum. Note that $c_ja_{ij}c_j^{-1}$ does not depend on our choice of c_i. Therefore this descent datum defines a canonical $\mathcal{G}|_U$-torsor $\widetilde{\mathcal{P}} \otimes \mathcal{T}$. Since $\pi(h_{ij}a'_{ij}) = \pi(h_{ij})$, we see that ϕ defines a morphism $\mathcal{P}|_U \to_{(\widetilde{\mathcal{P}} \otimes \mathcal{T})} \mathcal{G}_0$.

Suppose we have a morphism of descent data

$$(f_i) : (\mathcal{G}|_{U_i}, h_{ij}) \to (\mathcal{G}|_{U_i}, h'_{ij}), \ f_i \in \mathcal{G}(U_i)$$

corresponding to a morphism $(\widetilde{\mathcal{P}}, \phi) \to (\widetilde{\mathcal{P}}', \phi')$ in $\mathcal{Q}_{\mathcal{P}}(U)$. We refine the cover to assume $\phi_i = \pi(c_i)$ and $\phi'_i = \pi(c'_i)$ (notation as in the previous paragraph). Then from the definitions we have $h'_{ij}f_j = f_ih_{ij}$ and $\pi(f_ic_i) = \pi(c'_i)$. Thus

$$h'_{ij}c'_ja_{ij}c_j^{-1}f_j = h'_{ij}f_jc_ja_{ij}c_j^{-1} = f_ih_{ij}c_ja_{ij}c_j^{-1}.$$

Therefore (f_i) defines a morphism $(\widetilde{\mathcal{P}} \otimes \mathcal{T}, \phi) \to (\widetilde{\mathcal{P}}' \otimes \mathcal{T}, \phi')$.

Let $(b_i) : (\mathcal{A}|_{U_i}, \psi_{ij}) \to (\mathcal{A}|_{U_i}, \psi'_{ij})$, $b_i \in \mathcal{A}(U_i)$ be a morphism of descent data corresponding to a morphism $\mathcal{T} \to \mathcal{T}'$ in $\mathrm{Tors}(_\mathcal{P}\mathcal{A})(U)$. Here ψ_{ij}, ψ'_{ij} are defined with respect to $a_{ij}, a'_{ij} \in \mathcal{A}(U_{ij})$, respectively, as described above. The compatibility condition $\psi'_{ij}(b_j) = b_i\psi_{ij}(1)$ required for (b_i) to be a morphism of descent data is equivalent to

$$g_{ij}a'_{ij}b_jg_{ij}^{-1} = b_ig_{ij}a_{ij}g_{ij}^{-1} \iff h_{ij}c_ja'_{ij}c_j^{-1}c_jb_jc_j^{-1} = c_ib_ic_i^{-1}h_{ij}c_ja_{ij}c_j^{-1}.$$

Therefore $(c_ib_ic_i^{-1})$ defines a morphism $(\widetilde{\mathcal{P}} \otimes \mathcal{T}, \phi) \to (\widetilde{\mathcal{P}} \otimes \mathcal{T}', \phi)$.

We have now defined an action of $\mathrm{Tors}(_\mathcal{P}\mathcal{A})$ on $\mathcal{Q}_{\mathcal{P}}$. To check that $\mathcal{Q}_{\mathcal{P}}$ is a gerbe over $\mathrm{Tors}(_\mathcal{P}\mathcal{A})$, we show that for a fixed $(\widetilde{\mathcal{P}}, \phi) \in \mathcal{Q}_{\mathcal{P}}(U)$, the functor

$$\mathrm{Tors}(_\mathcal{P}\mathcal{A})(U) \to \mathcal{Q}_{\mathcal{P}}(U) : \mathcal{T} \mapsto (\widetilde{\mathcal{P}} \otimes \mathcal{T}, \phi)$$

is an equivalence of groupoids. We use the notation for descent data from the preceding paragraphs. First, we show fully faithfulness. Let $f_i \in \mathcal{G}(U_i)$ correspond to a morphism $(\widetilde{\mathcal{P}} \otimes \mathcal{T}, \phi) \to (\widetilde{\mathcal{P}} \otimes \mathcal{T}', \phi)$. The compatibility conditions require that $\pi(f_i) = \pi(1)$ and

$$h_{ij}c_ja'_{ij}c_j^{-1}f_j = f_ih_{ij}c_ja_{ij}c_j^{-1} \iff \psi'_{ij}(c_j^{-1}f_jc_j) = (c_i^{-1}f_ic_i)\psi_{ij}(1).$$

Therefore $b_i = c_i^{-1}f_ic_i \in \mathcal{A}(U_i)$ corresponds bijectively to a morphism $\mathcal{T} \to \mathcal{T}'$.

Next, let us show essential surjectivity of the functor. Take $(\widetilde{\mathcal{P}}', \phi') \in \mathcal{Q}_{\mathcal{P}}(U)$. We have $\pi(c_i^{-1}h_{ij}c_j) = \pi(c_i'^{-1}h'_{ij}c'_j) = g_{ij}$. Thus there exist unique $a_{ij} \in \mathcal{A}(U_{ij})$ such that

$$c_i'^{-1}h'_{ij}c'_j = c_i^{-1}h_{ij}c_ja_{ij}.$$

Since the h'_{ij} satisfy the cocycle condition, (6.2.1.1) shows that the transition morphisms ψ_{ij} associated to a_{ij} satisfy the cocycle condition. Therefore $(\mathcal{A}|_{U_i}, \psi_{ij})$ defines a descent datum, which corresponds to a torsor $\mathcal{T} \in \mathrm{Tors}(_\mathcal{P}\mathcal{A})(U)$, and $(c'_ic_i^{-1})$ corresponds to a morphism of

descent data for $(\widetilde{\mathcal{P}} \otimes \mathcal{T}, \phi) \to (\widetilde{\mathcal{P}}', \phi')$. Therefore the functor in question is an equivalence, and we conclude that $\mathcal{Q}_\mathcal{P}$ is a gerbe over $\mathrm{Tors}(_\mathcal{P}\mathcal{A})$. $\qquad\square$

6.3. **Proving smoothness.**

Proof of Proposition 6.0.18. Smoothness is local on S, so we may assume S is noetherian by Remark 4.1.4. We will prove $\mathrm{Bun}_G \to S$ is smooth by showing that it satisfies the conditions of Lemma 6.1.1. Let $T = \mathrm{Spec}\, A$ be an affine noetherian scheme and T_0 a closed subscheme defined by a square zero ideal $I \subset A$. Suppose we have a G-bundle $\mathcal{P}$ on X_{T_0} corresponding to a morphism $T_0 \to \mathrm{Bun}_G$. Since $X_{T_0} \to X_T$ is a homeomorphism of underlying topological spaces, [SGA1, I, Théorème 8.3] implies that base change gives an equivalence of categories between small étale sites

$$(X_T)_{\text{ét}} \to (X_{T_0})_{\text{ét}} : U \mapsto U_0 = U \underset{T}{\times} T_0.$$

Using this equivalence, we define two sheaves of groups $\mathcal{G}_0, \mathcal{G}$ on $(X_{T_0})_{\text{ét}}$ by

$$\mathcal{G}_0(U_0) = \mathrm{Hom}_k(U_0, G) \text{ and } \mathcal{G}(U_0) = \mathrm{Hom}_k(U, G).$$

Since G is smooth, infinitesimal lifting [BLR90, 2, Proposition 6] implies that $\mathcal{G} \to \mathcal{G}_0$ is a surjection of sheaves. Additionally, smoothness of G implies that $\mathcal{P}$ is étale locally trivial. Thus $\mathcal{P}$ is a $\mathcal{G}_0$-torsor on $(X_{T_0})_{\text{ét}}$, and a lift of $\mathcal{P}$ to a G-bundle on X_T is the same as a $\mathcal{G}$-torsor inducing $\mathcal{P}$. This is equivalent to an element of the lifting gerbe $\mathcal{Q}_\mathcal{P}(X_{T_0})$.

Consider the short exact sequence of sheaves of groups

$$1 \to \mathcal{A} \to \mathcal{G} \to \mathcal{G}_0 \to 1.$$

We give an explicit description of $\mathcal{A}$. Suppose $U = \mathrm{Spec}\, B$ is affine and $U_0 = \mathrm{Spec}\, B/IB$ for an A-algebra B. We will use the notation $k[G] = \Gamma(G, \mathcal{O}_G)$. Let $\varepsilon : k[G] \to k$ be the morphism of k-algebras corresponding to $1 \in G(k)$. Then $\mathcal{A}(U_0)$ consists of the $\varphi \in \mathrm{Hom}_k(k[G], B)$ such that the compositions $k[G] \overset{\varphi}{\to} B \to B/IB$ and $k[G] \overset{\varepsilon}{\to} k \hookrightarrow B/IB$ coincide. Define the morphism $\delta = \varphi - \varepsilon : k[G] \to IB$ of k-modules. Observe that for $x, y \in k[G]$, we have $\delta(x)\delta(y) = 0$ since $I^2 = 0$. Thus

$$\varphi(xy) - \varphi(x)\varphi(y) = \delta(xy) - \varepsilon(x)\delta(y) - \varepsilon(y)\delta(x) = 0,$$

which implies that φ corresponds bijectively to a derivation

$$\delta \in \mathrm{Der}_k(k[G], IB) \simeq \mathrm{Hom}_k(\mathfrak{m}_1/\mathfrak{m}_1^2, IB),$$

where IB is a $k[G]$-module via ε and $\mathfrak{m}_1 = \ker \varepsilon$ is the maximal ideal. Since $U \to X_T \to T$ is flat, B is a A-flat. Therefore $IB = I \otimes_A B$. Noting that $\mathfrak{m}_1/\mathfrak{m}_1^2 = \mathfrak{g}^\vee$, we have

$$\mathcal{A}(U_0) \simeq \mathrm{Hom}_k(\mathfrak{g}^\vee, IB) \simeq (\mathfrak{g} \underset{k}{\otimes} I) \underset{A}{\otimes} B \simeq (\mathfrak{g} \underset{k}{\otimes} I) \underset{A/I}{\otimes} B/IB$$

since $I^2 = 0$ makes I into an A/I-module. Let $\mathcal{I}$ be the ideal sheaf corresponding to $X_{T_0} \hookrightarrow X_T$, which is coherent because we assume T is noetherian. We have shown that $\mathcal{A}$ is the abelian sheaf corresponding to $\mathfrak{g} \otimes_k \mathcal{I} \in \mathrm{Coh}(X_{T_0})$ (see [Sta, Lemma 03DT]). Since $\mathcal{G}_0$ acts on $\mathcal{A}$ by conjugation, we see from the above isomorphisms that $\mathcal{G}_0$ acts on $\mathfrak{g} \otimes_k \mathcal{I}$ via the adjoint representation on $\mathfrak{g}$. Therefore

$$_\mathcal{P}\mathcal{A} \simeq {}_\mathcal{P}(\mathfrak{g}) \underset{\mathcal{O}_{X_{T_0}}}{\otimes} \mathcal{I},$$

which is a coherent sheaf by quasi-coherent descent [FGI$^+$05, Theorem 4.2.3] and persistence of finite presentation under fpqc morphisms [EGAIV2, Proposition 2.5.2].

There is a bijection between equivalence classes of gerbes over $\mathrm{Tors}(_{\mathcal{P}}\mathcal{A})$ and the étale cohomology group $H^2((X_{T_0})_{\text{ét}}, _{\mathcal{P}}\mathcal{A})$ by [Gir71, Théorème 3.4.2]. Since $_{\mathcal{P}}\mathcal{A}$ is in fact quasi-coherent, [SGA4-2, VII, Proposition 4.3] gives an isomorphism

$$H^2(X_{T_0}, _{\mathcal{P}}\mathcal{A}) \simeq H^2((X_{T_0})_{\text{ét}}, _{\mathcal{P}}\mathcal{A}),$$

where the left hand side is cohomology in the Zariski topology. By assumption, the fibers of $X \to S$ are of dimension 1. It follows from [EGAIV2, Corollaire 4.1.4] that $X_{T_0} \to T_0$ also has fibers of dimension 1. Now since $_{\mathcal{P}}\mathcal{A}$ is coherent, [Tôhoku, Théoreme 3.6.5] and [FGI$^+$05, 8.5.18] imply that $H^2(X_{T_0}, _{\mathcal{P}}\mathcal{A}) = 0$. Therefore by Lemma 6.2.1, the lifting gerbe $\mathcal{Q}_{\mathcal{P}}$ is neutral, i.e., $\mathcal{Q}_{\mathcal{P}}(X_{T_0})$ is nonempty. $\square$

Example 6.3.1. We mention that Proposition 6.0.18 need not be true if $X \to S$ has fibers of higher dimension. If $S = \mathrm{Spec}\, k$ is the spectrum of an algebraically closed field of positive characteristic, [Igu55] shows that there exists a smooth surface X over k such that $\mathrm{Pic}_{X/k}$ is not reduced, and hence not smooth. Corollary 4.2.6 gives a smooth surjective morphism of algebraic stacks $\mathrm{Bun}_1 \to \mathrm{Pic}_{X/S}$. Since the property of smoothness is local on the source in the smooth topology [EGAIV4, Théorème 17.11.1], we conclude that Bun_1 is not smooth over S.

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